Asymptotic analysis of the evolutionary snowdrift game on a cycle

by

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Abstract

Cooperation abounds in the natural world. Behaviour transcending selfishness has been witnessed among humans and in the animal kingdom throughout history. The underlying principles of this cooperation have become a focal point of study in the field of evolutionary game theory. The *snowdrift game* is a social dilemma in the form of a 2-player, 2-strategy game which has been used within this field in attempts to understand the precise nature of cooperation.

The concept of population structure is employed in the field of evolutionary spatial game theory in attempts at investigating the occurrence and subsequent persistence of cooperation in competitive environments. Players are modelled as the vertices of a graph, representing structure amongst the players, in which pairs of players iteratively play games against each other over successive rounds if their corresponding vertices are adjacent in the graph structure. Adopting the basic learning assumption that players adopt playing strategies that mimic the best performing players in their neighbourhoods, the temporal dynamics of the (deterministic) *evolutionary spatial snowdrift game* (ESS) can be investigated.

The results of such an investigation are documented in this thesis. After adapting an existing mathematical model for analysing the temporal dynamics of another evolutionary spatial game to the context of the ESS, a similar analysis is conducted for the ESS played on cycle graphs.

The investigation is conducted within the context of three game parametric regions in which the temporal game dynamics differ significantly from one another. For each region, the probability of persistent cooperation is determined. This requires a complete characterisation of randomly generated initial game states which lead to persistent cooperation. Bounds are also established on the fixation probabilities of the two strategies of the ESS, namely the strategies of cooperation and defection, and the relative magnitudes of these probabilities are compared for each of the three aforementioned regions. Finally, the components of the ESS state graph, which captures all possible temporal dynamics of the ESS graphically, are enumerated in each of the parametric regions.

In general, it is found that the probability of persistent cooperation increases with the order of the underlying cycle. Furthermore, in two of the three parametric regions, the strategy of cooperation is favoured above the strategy of defection, supporting the hypothesis that the strategy of cooperation dominates in the ESS played on cycles.

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Opsomming

Samewerking is volop in die natuur. Gedrag wat selfsug die hoof bied, is deur die geskiedenis heen onder mense en in die diereryk waargeneem. Die beginsels onderliggend aan hierdie samewerking het 'n fokuspunt van studie in die gebied van die evolusionêre spelteorie geword. *Sneeudrif* is 'n sosiale dilemma in die vorm van 'n 2-speler, 2-strategie spel wat in hierdie veld gebruik word om die presiese aard van samewerking te probeer verstaan.

Die konsep van bevolkingstruktuur word in the studieveld van evolusionêr-ruimtelike speleteorie gebruik in pogings om die voorkoms en daaropvolgende volharding van samewerking in mededingende omgewings te ondersoek. Spelers word as die punte van 'n grafiek gemodelleer, wat struktuur onder die spelers voorstel, waarin pare spelers iteratief spele in opeenvolgende rondtes teen mekaar speel as hul ooreenstemmende punte in die grafiekstruktuur naasliggend is. Gebaseer op die basiese leer-aanname dat spelers spelstrategiee aanneem wat die beste spelers in hul onmidellike omgewings naboots, kan die temporele dinamika van die (deterministiese) *evolusionêr-ruimtelike sneeudrifspel* (ERS) ondersoek word.

Die resultate van só 'n ondersoek word in hierdie tesis gedokumenteer. Nadat 'n bestaande wiskundige model vir die analise van die temporele dinamika van 'n ander evolusionêr-ruimtelike spel aangepas is vir die konteks van die ERS, word 'n soortgelyke analise geloots vir die ERS wat op sikliese grafieke gespeel word.

Die ondersoek vind binne die konteks van drie parametriese spelgebiede plaas, waarin die temporele speldinamika noemenswaardig van mekaar verskil. Vir elke gebied word die waarskynlikheid van volgehoue samewerking bepaal. Hierdie berekening vereis 'n volledige karakterisering van kans-gegenereerde aanvanklike speltoestande wat tot volgehoue samewerking lei. Grense word ook op die fiksasie-waarskynlikhede van die twee strategieë van die ERS daargestel, naamlik die strategieë van samewerking en afwyking, en die relatiewe groottes van hierdie waarskynlikhede word vir elk van die drie bogenoemde gebiede met mekaar vergelyk. Laastens word die komponente van die ERS-toestandsgrafiek, wat elke moontlike temporele dinamika van die ERS grafies vaslê, in elk van die parametriese gebiede getel.

In die algemeen word bevind dat die waarskynlikheid van volgehoue samewerking toeneem namate die orde van die onderliggende siklus toeneem. Hierbenewens word die strategie van samewerking in twee van die drie parametriese gebiede bo dié van die strategie van afwyking bevoordeel, wat die hipotese ondersteun dat die strategie van samewerking dié van afwyking in die ERS op siklusse oorheers.

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List of Acronyms

ESPD: Evolutionary Spatial Prisoner's Dilemma

ESS: Evolutionary Spatial Snowdrift Game

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CHAPTER 1

Introduction

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1.1 Background

The story of human history is littered with instances of cooperation. Before the rise of agriculture, bands of nomadic hunter gatherers cooperated when hunting large mammals. With the rise of agriculture, the cooperation of specialised workers facilitated the development of new technologies and of society in general. It is the cooperation of billions of humans that today allows for the exchange of currency in return for goods on a daily basis, the spread of knowledge through publication and communication, and the *mostly* peaceful journey through day-to-day life. While it is undoubtedly evident that humans have their fair share of quarrels and disputes, it may be even more ubiquitous that humans excel at cooperating with one another.

Considering Darwin's theory of natural selection and the struggle for existence, the realisation of cooperation within and between species becomes more and more intriguing. Where the competitive nature of the environment should encourage competition, cooperation is observed instead. One example of this phenomenon is sales clerks advising one on how to save money at their own establishments, when they could simply have allowed one to purchase foolishly, benefiting their own institutions and thereby enriching themselves. Another is the agreement of minimum prices for a product between firms, when a decreased price would instead offer a considerable increase in market share for the firm that decides to follow that route.

The quest to understand the persistence of cooperation in competitive settings has been the focus in various fields of study, from biology to sociology. One of the dynamics involved in the evolution of ideas occurs when observers identify a "successful" idea and adopt it for implementation. Survival of the fittest dictates that, over the course of many generations, the best ideas should persist while those with little merit should die out. Although various theories have attempted to explain it, a clear understanding of the survival of the strategy of cooperation in life remains somewhat elusive. Direct reciprocity, indirect reciprocity, graph selection, group selection and

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kin selection are a few of these theories aimed at explaining the persistence of cooperation, as discussed by Nowak in his influential book *SuperCooperators* [20].

Direct reciprocity employs the notion of "tit-for-tat" to explain how cooperation may persist. The idea is quite simply that an individual will help another in need, hoping to be helped in the future by that same individual when an own need arises [34]. The risk involved in lending a helping hand should be relatively low compared with the gains obtained when helped. This theory clearly depends on continued interaction as a favour can only be returned if there are future interactions.

Indirect reciprocity is abstracted one level higher, employing the idea of a reputation to explain why individuals might exhibit seemingly selfless behaviour [19]. If a person is known to have helped others in the past, then third parties might be inclined to help that person in the future. This relies on the image of the individual in need of help which, in turn, depends on the individual having been altruistic in the past. Indirect reciprocity employs the continued interaction of members in a society and their sharing of information to explain why it may be advantageous to cooperate at a cost to oneself.

The theory of kin selection posits that related individuals are more likely to behave altruistically when interacting with one another [8]. Formally, it describes the necessary conditions for an individual to help another in terms of the cost incurred to the cooperator, the benefit received, and the degree of genetic relation between the two. The key requirement is that individuals need to be related more closely as the cost to benefit ratio increases. Hamilton [8] used this theory to explain why birds might warn each other of approaching predators as a result of some shared genetic material in combination with the cost-to-benefit ratio of the act.

The theory of group selection has experienced rises and falls in popularity over time. One relatively recent study by Traulsen and Nowak [33] states that groups within a species possessing an altruistic gene are likely to exhibit a higher level of fitness than groups without such a gene and therefore selection will propagate those groups possessing the altruistic gene, while the groups without cooperators present will die out. This has been explained by investigating a game between members of a group affecting the fitness of the individuals involved and thereby the overall fitness of the group. Nowak [20] also claimed that this form of selection may be linked quite strongly to the indirect reciprocity theory of persistent cooperation.

The last theory of cooperation is that of graph selection [23], which focuses on the structure of interactions by the players of a game. Basic notions in graph theory may be used to model these structured interactions and to investigate which structures might opt for (or against) cooperation. This type of investigation has gained traction in recent years and is also the nature of the investigation documented in this thesis.

Methods for investigating the effects of graph (network) structure often involve simulation methods due to the combinatorial complexity of game dynamics on these graphs. The investigation in this thesis, however, involves the application of analytic methods on the simple graph structure of a cycle. The topic of investigation is the emergence of persistent cooperation in the social dilemma of the well-known *snowdrift game* in game theory. One of the questions of interest is whether the strategy of cooperation has a chance of persistence in the particular setting of the *evolutionary spatial snowdrift game* (ESS) on a cycle. An investigation is conducted into the long-time asymptotic behaviour of players of the game as a function of the size of the underlying player population.

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1.2 Problem statement and research questions

This thesis is a record of an investigation into the ESS played on cycle graphs. Players of the game are assumed to "learn" by mimicking better performing neighbouring players in a deterministic setting. The game is played in a series of successive rounds, each player can adopt either the strategy of cooperation or defection during any round, and the key question is whether or not the strategy of cooperation is able to persist asymptotically in the long run. This question is answered by characterising the game dynamics theoretically. In order to do this, the notions of a fixation probability and game state, as well as the state graph of the ESS, are required.

The *fixation probability* of a strategy in a game is the probability that, once introduced into a population of players adopting another strategy, the introduced strategy comes to dominate within the population (all the players in the population eventually adopt the newly introduced strategy). The game rounds are characterised in terms of game states. A *game state* is an assignment of strategies to the players in the population. Finally, a *state graph* is a graphical representation of the game dynamics in which all possible game states are represented as vertices and in which arcs (directed edges) indicate which game state leads to and from each other game state.

The specific research questions addressed in this thesis are:

1. Which initial conditions guarantee that the strategy of cooperation persists indefinitely in some form among the population of players?

2. How does the size of the cycle structure underlying the game influence the asymptotic behaviour of players in the game in terms of the persistence of cooperation?

3. Can the notion of a fixation probability be applied to the deterministic setting of the game in some way and, if so, which strategy is favoured in the ESS on a cycle?

4. How many components are there in the state graph of the game?

All of these questions are investigated in the context of the ESS on a cycle within various pay-off parameter regions of the phase plane of the game in which player behaviour is expected to be distinct.

1.3 Research scope

The analysis undertaken in this thesis is limited in a variety of aspects as a result of time constraints, restrictions as a result of analysis complexity, and the particular area of investigation. The general idea is to investigate the spatial evolution of the strategy of cooperation in the context of the ESS, a social dilemma game, played on a cycle.

The player population structure under investigation is limited to cycle graphs. This holds both for the player interaction graph and the learning graph, which are assumed to be identical throughout. The reason for this scope delimitation is to facilitate analytical tractability of the ensuing investigation as computer simulation is to be avoided. Although other regular graphs may also prove to be analytically tractable in this respect, such as circulant and ladder graphs, these are not considered due to time constraints.

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The selection dynamics considered in the area of evolutionary spatial game theory often include some form of stochasticity during the process of choosing an individual which is to die or reproduce. Furthermore, stochasticity in the learning action of individuals adopting new strategies is also often observed in the literature. These dynamics may be considered individualistic, and constitute weak selection. In contrast, the dynamics considered in this thesis are deterministic and global. Eliminating stochasticity aids in the general analytical tractability and makes for a clearer investigation into populations of players organised along cycles in which each individual has the opportunity to update its strategy and does so deterministically based on the performance of its closed neighbourhood of players.

There are a host of 2-player, 2-strategy games that can be investigated in the setting described thus far. Due to a gap in the literature, however, and the limited time available for the investigation in this thesis, the research scope is limited to the snowdrift game (in an evolutionary setting) and its particular pay-off parameter inequality chain, assuming only the pure strategies of cooperation and defection.

The limitations described above allow for the specific requirements of persistent cooperation to be identified and exploited in order to determine the probability thereof. The nature of the investigation in this thesis focuses on steady state player behaviour, taking into consideration the order of the state graph of the game, as well as the initial conditions that allow for the strategy of cooperation to dominate. The investigation conducted in this thesis is an extension of the work of Burger *et al.* [4] and Van der Merwe [35] on the *evolutionary spatial prisoner's dilemma* (ESPD). The work also contains an extension in that the notion of a fixation probability (which has been studied in the literature on evolutionary spatial games) is pursued.

1.4 Study objectives

The following objectives are pursued in this thesis:

I To *conduct* a survey of the literature related to the ESS with special attention afforded to evolutionary games on cycles.

II To *revise* an existing mathematical framework for identifying structures and patterns within the assignment of strategies to players of evolutionary games on cycle graphs which give rise to cooperation.

III To *determine* the likelihood for persistent cooperation in the ESS on cycles, given randomly generated initial player strategies.

IV To *extend* the notion of a fixation probability within the context of deterministic, synchronous games and to establish bounds on this probability within the context of the game studied.

V To *establish* bounds on the number of components in the state graph of the ESS on a cycle.

1.5 Thesis organisation

In Chapter 2, a variety of mathematical concepts which are used in the remainder of the thesis are defined and discussed. The purpose of the chapter is to introduce relevant concepts that

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serve as a foundation for the arguments made in the subsequent analysis in the interest of self-containment of the discourse. The chapter opens with a brief overview of basic notions in graph theory, as well as groups of special classes of graphs employed in the thesis either as the subject of investigation or as tools to aid in the investigation to follow. In particular, the transfer matrix method is discussed and so is the underlying notion of a generating function. Lastly, in an effort to elucidate the Cauchy-Frobenius Lemma, basic notions in group theory and group actions are also discussed.

Chapter 3 contains a review of the literature relevant to the analysis in this thesis. It opens with a discussion on notions and important developments in classical game theory and includes an introduction to two widely studied games, the prisoner's dilemma and the snowdrift game. The discussion then evolves towards iterated games, an important development in the study of game theory. Evolutionary spatial and graphical games are also considered, both of which add a population structure to the dynamics previously described in the context of a well-mixed population. The chapter closes with a discussion on recent work in evolutionary games on cycles.

The focus of Chapter 4 falls on the mathematical representation of the ESS. The concepts of Chapters 2 and 3 are tied together in order to represent the ESS on a cycle in such a way so as to aid in the investigation thereof. A normalisation of the pay-off parameter values is carried out, which results in a pay-off matrix containing only two parameters, thus paving the way for a succinct representation of game parameters in a two-dimensional phase plane in which three regions of differing game dynamics are delineated. Finally, the concept of a fixation probability is discussed, including a description of a proposed variation thereof which expands the reach of the concept to the deterministic setting of this thesis.

The dynamics of the ESS on a cycle in the first phase plane region of interest are considered in Chapter 5, initially with respect to its relation to the ESPD on a cycle and subsequently drawing conclusions from this relation. The topics covered are the requirements for and the probability of persistent cooperation from a randomly generated initial state, both of which are enumerated precisely. The discussion thereafter shifts to a brief investigation into a variation on the notion of a fixation probability for the strategies of cooperation and defection, respectively, leading to the conclusion that the strategy of defection is favoured in this phase plane region of the ESS on a cycle. Finally, the number of components in the state graph of the ESS on a cycle in the first phase plane region is enumerated.

Chapter 6 contains a discourse on the same topics as those considered in Chapter 5, but this time for the second phase plane region. The chapter opens with the establishment of preliminary results which aid the investigation in the remainder of the chapter. Thereafter, the requirements for and probability of persistent cooperation are determined for ESS on a cycle in the particular region of the phase plane. The investigation then turns again to the notion of a fixation probability, this time leading to the conclusion that the strategy of cooperation is favoured. The chapter closes with the establishment of a lower bound on the number of components in the state graph.

The final analysis chapter, Chapter 7, contains the results of the investigation for the third phase plane region of interest. These results again include the requirements for and probability of persistent cooperation from a randomly generated initial distribution of strategies, in the context of this particular parameter region. The notion of a fixation probability is then applied to the strategies of cooperation and defection in the context of this region, leading to the conclusion that the strategy of cooperation is favoured over that of defection. Lower bounds on the number of components in the state graph are established in closing.

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The final chapter contains a discussion on the conclusions that can be drawn from the study documented in this thesis as well as a self-appraisal of the contributions of the thesis. Possible future work is also suggested in this final chapter. This future work includes possible improvements on bounds enumerated in the thesis by adopting a more intricate enumeration procedure, as well as alternative investigations that follow naturally from the results of this study. These continuing investigations include consideration of underlying graph structures other than cycles, differing update rules and 2-player, 2-strategy games other than the ESS and the ESPD.

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CHAPTER 2

Preliminary concepts and methods

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The work documented in this thesis requires an understanding of certain basic mathematical principles, which are elucidated in this chapter. The chapter opens in §2.1 with a brief introduction to a number of basic definitions from the realm of graph theory. These definitions are illustrated by means of simple examples in each chase. Thereafter, a number of the special infinite classes of graphs that are relevant in the context of this thesis are discussed in §2.2. Two well-known tools in enumerative combinatorics are reviewed thereafter. The first, the transfer matrix method, is described in §2.3, while the second, the Cauchy-Frobenius Lemma, is discussed and illustrated in §2.4. The notational conventions adopted in this chapter conform to those in [11], [31] and [15] for the topics in graph theory, the transfer matrix method and the Cauchy-Frobenius Lemma, respectively. The chapter finally closes with a summary of its contents in §2.5

2.1 Graph theoretic basics

A graph G is a non-empty, finite set of *vertices* or *nodes* in combination with another possibly empty and finite set of *edges* between these vertices (represented by unordered pairs of vertices). These sets are denoted by $V(G)$ and $E(G)$, respectively. The number of vertices in G is known

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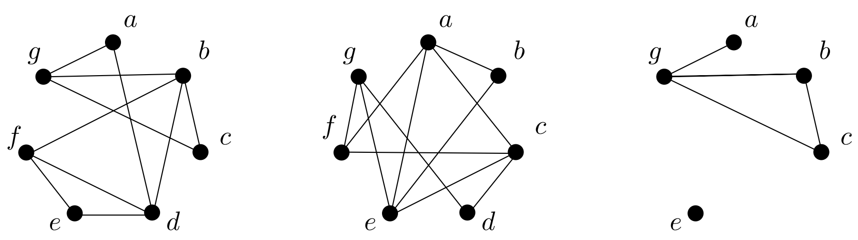

(a) G_1 (b) $\overline{G}_1$ (c) H_1

FIGURE 2.1: *The graph G_1, its complement $\overline{G}_1$ and a possible subgraph, $H \subseteq G$.*

as its *order* and is denoted by $n(G)$ or simply n, while the number of edges in G is known as its *size* and is denoted by $m(G)$ or m.

In graphical representations of graphs vertices are depicted as points, or small circles, while edges are depicted as lines joining pairs of vertices. An edge $e = \{u, v\} = uv$ is therefore represented by a line that joins the vertices u and v. The presence of an edge between two vertices makes these vertices *adjacent* to one another (in which case they are called *neighbours*). The edge e is *incident* to the two vertices it joins (u and v in this case). Furthermore, two edges are called *adjacent* if they have a shared vertex. The assignment of colours to each of the vertices in a graph is called a *graph colouring* which is distinct from *vertex colouring* which includes the limitation that adjacent vertices may not be of the same colour.

The *complement* of a graph G, denoted by $\overline{G}$, is the graph with the same vertex set as G but in which two vertices are joined by an edge if and only if they are not adjacent in G. A graph G_1 and its complement $\overline{G}_1$ are represented graphically in Figures 2.1(a) and 2.1(b), respectively. A *subgraph* of G is a graph generated by any subset of the vertices within G (possibly all of them, in which case it is called a *spanning* subgraph) and including edges between any pairs of vertices that are adjacent in G. If H is a subgraph of G, this is denoted by writing $H \subseteq G$. In Figure 2.1(c), H_1 is a subgraph of the graph G_1 in Figure 2.1(a).

The *degree* of a vertex u in a graph G, denoted by $d_G(u)$, is the number of edges with which it is incident. For any graph G, the largest degree of a vertex in the graph is denoted by $\Delta(G)$, while the smallest degree of a vertex in G is denoted by $\delta(G)$. When the graph in question is clear from the context these notations are simplified to $d(u)$, Δ and δ, respectively. Vertices of degree 0 are known as *isolated* vertices. The parity (*odd* or *even*) of the degree of a vertex is the parity of the vertex itself, with the convention that 0 is even. A vertex of degree of 1 is known as an *end-vertex*.

Each vertex v in a graph is associated with an *open* and a *closed neighbourhood*. The first consists of all the vertices adjacent to the vertex v, and is denoted by $N(v)$, while the second is denoted by $N[v]$, and is given by $N[v] = N(v) \cup \{v\}$.

The *degree distribution* of a graph, denoted by $D(k)$, is a function which yields the probability of a randomly selected vertex having degree k within that graph. The degree distribution of the graph G_1 in Figure 2.1(a) is

$$D(k) = \begin{cases} \frac{3}{7} & \text{if } k = 2, \\ \frac{2}{7} & \text{if } k = 3, \\ \frac{2}{7} & \text{if } k = 4, \\ 0 & \text{otherwise.} \end{cases}$$

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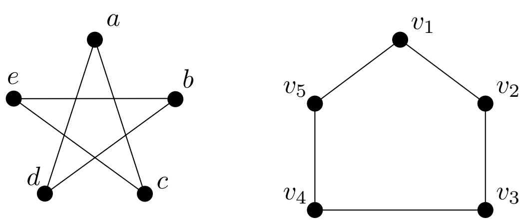
2.1.1 Isomorphisms

Graphs that are isomorphic to one another have a shared structure, with differences occurring only in the labelling of the vertices. The structure of a graph does not depend on its layout in a graphical representation of the graph. Graphs that may at first appear to have different structures, may indeed turn out to be isomorphic or equivalent to one another. Formally, a graph G is *isomorphic* to a graph H (denoted by $H \cong G$) if there exists a bijective function $\phi : V(G) \mapsto V(H)$, called an *isomorphism*, such that uv is an edge of G whenever $\phi(u)\phi(v)$ is an edge of H. Figure 2.2 contains graphical illustrations of two isomorphic graphs. An isomorphism $\phi : V(G_2) \mapsto V(G_3)$ for the graphs in Figure 2.2 is defined as follows: $\phi(a) = v_1$, $\phi(b) = v_3$, $\phi(c) = v_2$, $\phi(d) = v_5$, and $\phi(e) = v_4$.

An isomorpism from the vertex set of a graph to itself is called an *automorphism*. An example of an automorphism $\phi : V(G_3) \mapsto V(G_3)$ for the graph in Figure 2.2(b) is defined as follows: $\phi(v_1) = v_2$, $\phi(v_2) = v_3$, $\phi(v_3) = v_4$, $\phi(v_4) = v_5$, and $\phi(v_5) = v_1$.



(a) G_2 (b) G_3

FIGURE 2.2: *An example of two isomorphic graphs.*

2.1.2 Walks and connectivity

A *walk* in a graph is an ordered set of vertices which have the property that each successive pair of vertices is joined by an edge. Such a walk may be thought of as a route through the graph traversed along successive pairwise adjacent edges. The length of a walk from u to v (called a *u-v walk*) is the number of edges traversed along the walk. A walk in which no vertices are repeated is called a *path*, and a walk which starts and ends at the same vertex is called a *closed walk*. A closed walk in which no vertex, except the first and last, is repeated is called a *cycle*. A cycle of length k is called a *k-cycle* or a *cycle of order k*. In the graph G_4, in Figure 2.3, $v_2v_1v_4v_5v_6$ is an example of a v_2-v_6 walk of length 4, which is also a path, while v_2v_6 is an example of a v_2-v_6 walk of length 1. Moreover, $v_1v_2v_3v_1v_4v_1$ is an example of a closed walk which is not a cycle, while $v_1v_2v_3v_4v_1$ is an example of a 4-cycle or a cycle of order 4.

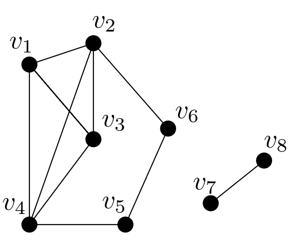


FIGURE 2.3: *The graph G_4.*

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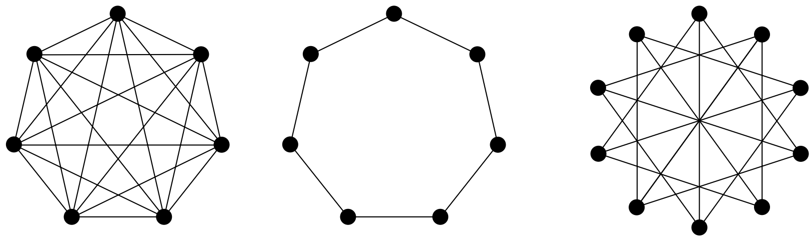
Graphs in which there are walks between all pairs of vertices are called *connected*. If there is at least one pair of vertices u and v for which there is no walk from u to v the graph is called *disconnected*. Disconnected graphs have at least two *components* which are defined as maximal connected subgraphs of the original graph. Note that there are no walks from v_2 to either v_7 or v_8 in the graph G_4 in Figure 2.3, because v_2 is in a different component than v_7 and v_8.

2.2 Special graphs

This section contains an elucidation of three infinite classes of graphs that play a central role later in this thesis. These graph classes are cycle graphs, complete graphs and circulants. The class of directed graphs is also discussed briefly in closing.

2.2.1 Regular graphs

A *regular graph* is a graph in which each vertex has the same degree. Two important regular graphs are the *complete graph*, in which each vertex is adjacent to every other vertex, and the *cycle graph*, in which each vertex is adjacent to two other vertices only and the graph is connected. The complete graph of order n is denoted by K_n, while the cycle graph of order n is denoted by C_n. The complete graph K_7 and the cycle graph C_7 are illustrated graphically in Figures 2.4(a) and 2.4(a), respectively.



(a) K_7 (b) C_7 (c) $C_{10}\langle 3, 5\rangle$

FIGURE 2.4: *Three examples of regular graphs. (a) The complete graph K_7, (b) the 7-cycle C_7 and (c) the circulant $C_{10}\langle 3, 5\rangle$.*

The *circulant graph* $C_n\langle k_1, \ldots, k_x\rangle$ has vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{v_i v_{(i+j) \bmod n} \mid i \in \{1, \ldots, n\}$ and $j \in \{k_1, \ldots, k_x\}\}$. The essence of constructing edges in this way is that every $k_1, \ldots, k_x$ vertices are joined to one another by edges when arranged along the edge of an imaginary circle. For example, the circulant $C_{10}\langle 3, 5\rangle$ is shown graphically in Figure 2.4(c).

When $x = 1$ and $k_x = 1$, the resulting circulant graph is simply the n-cycle. On the other hand, when $x = \lceil \frac{n}{2} \rceil$ and $k_1, \ldots, k_x = 1, 2, 3, \ldots, \lceil \frac{n}{2} \rceil$, the resulting circulant graph is the complete graph K_n. Note that the two regular graphs in Figures 2.4(a) and 2.4(b) may therefore also be considered circulants.

2.2.2 Digraphs and pseudodigraphs

A *directed graph*, or *digraph*, is a graph in which each edge, called an *arc*, is associated with a direction, and these directions are indicated by means of arrows in graphical representations. The *arc set* of a digraph is therefore a set of ordered pairs selected from its vertex set. An arc of

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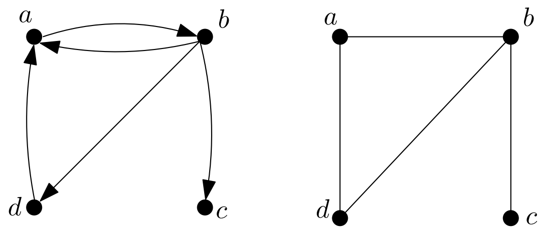
the form (u, v) is interpreted as being directed from the vertex u to the vertex v. An ordinary graph G (on the same vertex set) is associated with each digraph D and is called the *underlying graph* of D. This underlying graph is obtained from D by removing all directions from the arcs of D and by deleting an edge from each pair of repeated edges, should such multiple edges be produced. The underlying graph G_5 of the digraph D_1 in Figure 2.5(a) is, for example, shown in Figure 2.5(b).



(a) D_1 (b) G_5

FIGURE 2.5: *A digraph D_1 and its underlying graph G_5.*

Basic notions, such as order, size and walks, defined in the opening section of this chapter for graphs, may be extended naturally to digraphs by replacing the former notion of an edge with that of an arc (a directed edge), and making small changes in the definition, where necessary. For example, if (u, v) is an arc in a digraph D, then u is adjacent *to* v, while v is adjacent *from* u. Moreover, the arc (u, v) is incident from u while it is incident *to* v. A distinction is also made between the so-called indegree and the outdegree of a vertex in a digraph. The *outdegree* of a vertex v in a digraph D is denoted by $\mathrm{od}_D(v)$, or merely $\mathrm{od}(v)$, and is the number of vertices adjacent from v. Similarly, the *indegree* of v is denoted by $\mathrm{id}_D(v)$, or merely $\mathrm{id}(v)$, and is the number of vertices adjacent to v. The *degree* of a vertex v is denoted by $\mathrm{d}_D(v)$ or merely $\mathrm{d}(v)$ and is defined as $\mathrm{d}(v) = \mathrm{id}(v) + \mathrm{od}(v)$. The indegrees, outdegrees and degrees of the vertices of the digraph D_1 in Figure 2.5(a) are listed in Table 2.1. An arc joining a vertex to itself is known as a *loop*. A digraph with loops is known as a *pseudodigraph*.

TABLE 2.1: *The indegrees, outdegrees and degrees of the vertices of the digraph D_1 in Figure 2.5(a).*

v	$\mathrm{id}(v)$	$\mathrm{od}(v)$	$\mathrm{d}(v)$
a	2	1	3
b	1	2	3
c	1	0	1
d	1	1	2

2.3 The transfer matrix method

The need to enumerate closed walks in digraphs arises at various points later in this thesis. An efficient method whereby such an enumeration can be achieved is the *transfer matrix method*. This method makes use of the notion of a generating function. This section is devoted to a brief review of these two concepts.

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2.3.1 The notion of a generating function

In a generating function $F(x)$, the letter x no longer represents a variable which assumes a value, but rather performs the function of a placeholder and is called the *indeterminate*. The exponents of the indeterminate are of more interest in combinatorics than the indeterminate itself. Any infinite sequence $s_1, s_2, s_3, \dots$ may be identified with a generating function of the form

$$F(x) = \sum_{i=0}^{\infty} s_i x^i = s_1 x + s_2 x^2 + s_3 x^3 + \cdots$$

In this power series in the indeterminate x, the coefficients are the values in the sequence and have combinatorial meaning in a variety of counting arguments.

2.3.2 Transfer matrix method

The transfer matrix method may be used to count the number of closed walks of a certain length in a digraph. This method is thoroughly described in [31]. At the heart of the method is the result of the following theorem which describes how the number of walks of specified length, starting at some vertex v_i and ending at some vertex v_j within a digraph, may be obtained from the adjacency matrix of the digraph and its powers.

Theorem 2.1 ([31, p. 573]).
Let $\boldsymbol{A}$ be the adjacency matrix of a digraph D. Then the number of walks of length ℓ, starting at vertex v_i and ending at vertex v_j may be found in the row i and column j of the matrix power $\boldsymbol{A}^\ell$ for any $\ell \in \mathbb{N}$.

This result is particularly useful as extended in the following theorem to provide a method for counting the number of closed walks of length n in a given digraph, utilising a generating function. The coefficients of this generating function will be used later in this thesis to seed the values of a recurrence relation for counting closed walks in a digraph for lengths $w \to \infty$.

Theorem 2.2 ([31, pp. 574–575]).
Suppose $\boldsymbol{A}$ is the adjacency matrix of a digraph D, and that there are $C_D(\ell)$ closed walks of length ℓ in D. Then

$$\sum_{\ell=1}^{\infty} C_D(\ell) x^\ell = \frac{xT'(x)}{T(x)}, \tag{2.1}$$

where $T(x) = \det(\boldsymbol{I} - x\boldsymbol{A})$ and $\boldsymbol{I}$ is the identity matrix of the same dimension as that of $\boldsymbol{A}$.

These two theorems may be applied as follows to determine the number of closed walks of any particular length in a digraph: Given the adjacency matrix $\boldsymbol{A}$ of the digraph, the determinant of $(\boldsymbol{I} - x\boldsymbol{A})$ is first computed using methods from linear algebra. This makes it possible to evaluate the function $\frac{xT'(x)}{T(x)}$ in terms of the zeroeth and first derivative of the aforementioned determinant. The Maclaurin expansion of this function yields the power series $\sum_{w=1}^{\infty} C_D(w)x^w$, the coefficients of which represent the number of closed walks of the various possible length in the digraph in question.

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2.4 The lemma that is not Burnside's

A method for counting the number of isomorphism classes in a combinatorial species is given by the Cauchy-Frobenius Lemma (often mislabelled Burnside's lemma). In order to understand the lemma, a very basic knowledge of groups is required.

2.4.1 The notion of a group

A group is a set of elements along with a binary operator that adheres to the following four properties:

1. Closure — the result of combining any two elements of the group by means of the binary operator is again an element of the group.

2. Associativity — different associations of binary operations on elements of the group do not affect the result.

3. Identity — there exists an element ι in the group which maps each element to itself under the binary operation.

4. Inverse — each element α in the group has an inverse element α^{-1} which, when combined by the binary operator, yield the identity element ι of the group.

The *Cayley table* of a group is a tabular representation of the results obtained upon combining the various elements of a group by means of its binary operator. In particular, the entry in row i and column j of the Cayley table of a group contains the group element obtained by combining element i with element j.

The set $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ together with the binary operator $\odot$ of addition modulo n is a well-known example of a group. This may be seen by considering the Cayley table of the group $(\mathbb{Z}_5, \odot)$, shown in Table 2.2. The closure property of $(\mathbb{Z}_5, \odot)$ is immediately apparent from its Cayley table. Moreover, consider the equality of group elements $(2 \odot 4) \odot 1 = 1 \odot 1 = 2$ and $2 \odot (4 \odot 1) = 2 \odot 0 = 2$ as an example of the property of associativity being satisfied by $(\mathbb{Z}_5, \odot)$. The identity property is furthermore satisfied as a result of the presence of the group element 0; it is clearly the case that $0 \odot \alpha = \alpha \odot 0 = \alpha$ for any $\alpha \in \mathbb{Z}_5$. Finally, because there is exactly one zero in each row and each column of the Cayley table of $(\mathbb{Z}_5, \odot)$, it follows that each element of $\mathbb{Z}_5$ has a unique inverse. In particular, the inverses of 0, 1, 2, 3 and 4 are the group elements 0, 4, 3, 2 and 1, respectively.

TABLE 2.2: *The Cayley table for the group* $(\mathbb{Z}_5, \odot)$.

$\mathbb{Z}_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

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2.4.2 Group actions

Saying that a group $\mathcal{Q}$, *acts* on a set $\mathcal{U}$ means that there is a mapping $\tau_q : \mathcal{U} \mapsto \mathcal{U}$ for each $q \in \mathcal{Q}$ [15]. Each element $q \in \mathcal{Q}$ is an *action* which acts on the elements of $\mathcal{U}$ and is associated with the mapping τ_q. Furthermore, it is required that one of these actions, ι, is the identity action which maps each element $u \in \mathcal{U}$ to itself, and that $\tau_{qr}(u) = \tau_q(\tau_r(u))$ for all $u \in \mathcal{U}$ and any $q, r \in \mathcal{Q}$. Lastly, the action q *fixes* an element $u \in \mathcal{U}$ if $\tau_q(u) = u$. The set $\mathcal{U}$ may be partitioned into isomorphism classes which contain the elements that can map to one another under actions of the group $\mathcal{Q}$.

Theorem 2.3 (Cauchy-Frobenius Lemma [15, p. 92]).
The number of isomorphism classes into which a set $\mathcal{U}$ is partitioned by the group $\mathcal{Q}$, acting on the set $\mathcal{U}$, is

$$\mathcal{E}_u = \frac{1}{|\mathcal{Q}|} \sum_{q \in \mathcal{Q}} |F_q|,$$

where $|F_q|$ is the number of elements fixed by the action q.

The theorem above is proved in [15, Lemmas 1–8, pp. 89–92].

A combinatorial *necklace* is a string of n characters from an alphabet of k letters (otherwise known as n *beads* of k *colours*), with rotations considered equivalent. *Bracelets* are necklaces in which reflections are also considered equivalent. Theorem 2.3 is illustrated in the context of counting isomorphism classes of combinatorial bracelets in 6 beads of 2 colours. Graphically the bracelets may also be represented as graph colourings of cycles using two colours. Figure 2.6 depicts all of the bracelets in 6 beads of 2 colours in their respective isomorphism classes as ground truth for the ensuing enumeration.

The group acting on a cycle of order 6 is the dihedral group $\mathcal{D}_6 = \{\iota, \rho, \rho^2, \rho^3, \rho^4, \rho^5, \sigma_1, \sigma_2, \sigma_3, \delta_1, \delta_2, \delta_3\}$. In this group, ι is the identity element which maps each vertex of the cycle onto itself. The action ρ^i rotates each vertex of the cycle i positions in the clockwise direction, while the action σ_j reflects the vertices of the cycle around the diametrical axis passing through the vertex j. Finally, the action δ_k reflects the vertices of the cycle about the diametrical axis passing midway between vertices k and $k+1$.

Each vertex is mapped to itself under the action ι, and so $|F_\iota| = 2^6$ in the notation of Theorem 2.3. The action ρ maps each vertex to the vertex on its right, and therefore all of the vertices must be the same colour for a colouring to be fixed by this action, resulting in the cardinality $|F_\rho| = 2$. The action ρ^2 maps the first vertex to the third vertex, the third vertex to the fifth vertex and the fifth vertex to the first vertex. Similarly the second vertex is mapped to the fourth vertex, the fourth vertex is mapped to the sixth vertex and the sixth vertex is mapped to the second vertex. This means that the first and the second vertex are free to be coloured in 2^2 ways, upon which the remaining vertex colours are determined, so that $|F_{\rho^2}| = 2^2$. Similar arguments lead to the values $|F_{\rho^3}| = 2^3$, $|F_{\rho^4}| = 2^2$, and $|F_{\rho^5}| = 2$. The σ actions each map two vertices to themselves (the ones on the axis of reflection), while the remaining four vertices map to one another in pairs. Therefore, there are four free vertices and two determined vertices, so that $|F_{\sigma_j}| = 2^4$ for $j = 1, 2, 3$. Finally, the δ actions maps the vertices to one another in pairs and hence there are three free vertices and three determined vertices. As a result, $|F_{\delta_k}| = 2^3$ for $k = 1, 2, 3$.

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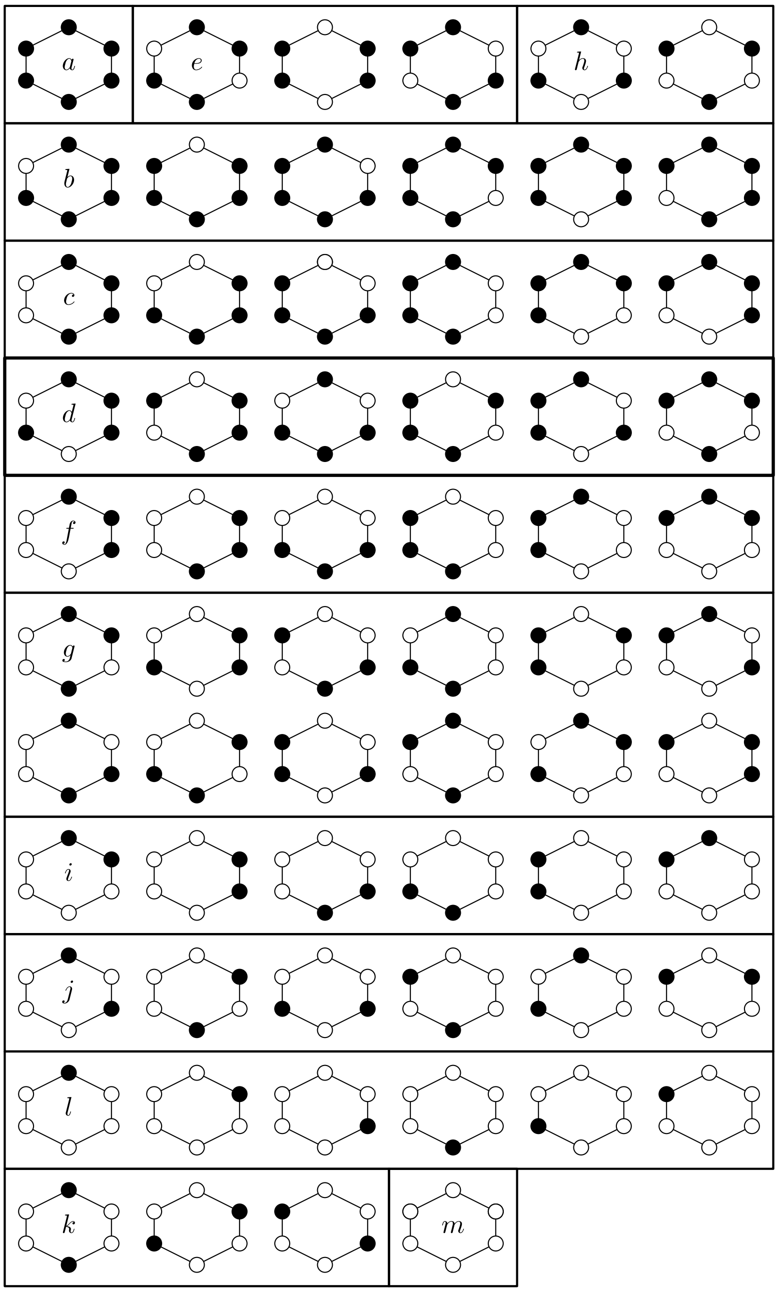

FIGURE 2.6: *Full isomorphism classes of bracelets in 6 beads of 2 colours with the labels in the class representatives corresponding to those in Figure 2.7.*

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Applying the Cauchy-Frobenius Lemma it therefore follows that the number of isomorphism classes of combinatorial bracelets in 6 beads of 2 colours is

$$\begin{aligned}|C_{6,2}| &= \frac{1}{|\mathcal{D}_6|}\sum_{d\in\mathcal{D}_6}|F_d| \\ &= \frac{1}{12}(2^6+2+2^2+2^3+2^2+2+2^4+2^4+2^4+2^3+2^3+2^3) = 13. \end{aligned} \tag{2.2}$$

These thirteen isomorphism classes are illustrated graphically in Figure 2.7, in which each class is assigned a label a–m.

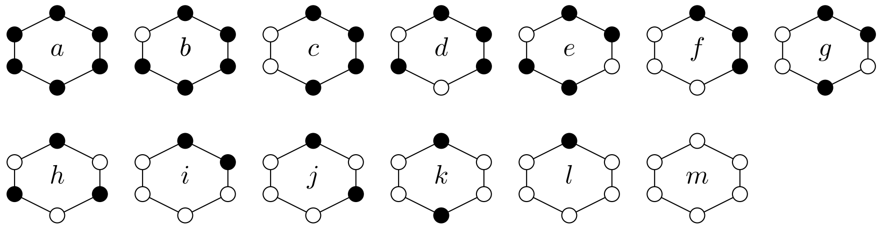

FIGURE 2.7: *Isomorphism classes of bracelets in 6 beads of 2 colours.*

The correctness of the value in (2.2) may be verified in Figure 2.6 in which all of the $2^6 = 64$ combinatorial bracelets in 6 beads of 2 colours are partitioned into thirteen isomorphism classes with the colourings in Figure 2.7 as representative isomorphism class members.

2.5 Chapter summary

This chapter was devoted to providing the reader with a brief introduction to a number of mathematical fundamentals of which a basic understanding is required in order to read this thesis. The chapter opened in §2.1 and §2.2 with reviews of a number of basic notions from the realm of graph theory and a description of a number of important graph classes, respectively. The transfer matrix method, which may be used to enumerate closed walks of a specified length in digraphs, was outlined in §2.3. The Cauchy-Frobenius Lemma (an important tool in enumerative combinatorics which is often mislabelled as Burnside's Lemma) was finally described and illustrated in §2.4.

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CHAPTER 3

Literature study

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In this chapter, the literature pertaining to game theory, evolutionary games, spatial games and games on cycles relevant to the topic of this thesis is reviewed. The chapter opens in §3.1 with a brief overview of basic notions in classical game theory, and this is followed by an introduction to two well-known 2×2 games, called the prisoner's dilemma and the snowdrift game, in §3.2. A brief review of iterated games, including Axelrod's computer tournament, is presented in §3.3, after which the focus shifts to spatial and graphical games in §3.4 and §3.5, respectively. The limited literature in the field of evolutionary games on cycles is finally reviewed in §3.6, after which the chapter closes in §3.7 with a summary of its contents.

3.1 Game theory

Game theory is the study of decision making in a multi-decision maker environment [6]. The decision makers constitute the players of the game, each of which selects a strategy from a strategy set and the combination of selected strategies determine the outcome for each player. The decision making is assumed to be purely rational and, as a result, it is assumed that each player plays the game with the goal of maximising his or her expected pay-off value. The objective in classical game theory is to find a solution which describes the strategy selection of each player of the game, given the rules of the game and the assumptions of rationality. There are a variety of methods for finding solutions, such as the minimax principle and the maximax principle, for example. The first of these involves players selecting their strategy by maximising the minimum pay-off value they can obtain, while the latter involves players selecting their strategy in order to maximise the maximum possible pay-off value they can obtain. Of course, these solutions are often not observed when actual players play the game, and they are not intended as predictors of such situations, but are rather descriptive of the nature of the game

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itself. The game considered in this thesis is part of the family of so-called non-cooperative games. In non-cooperative games, agreements, such as contracts between players, are not allowed and the players are considered to be in competition against one another.

Although games were studied before John von Neumann, he is considered the 'father' of game theory as his formalisation of the notion of a game brought about the onset of the study of games as a discipline in its own right. The minimax principle, proposed by von Neumann [37] for example, has become fundamental in the study of games. Furthermore, von Neuman and Morgenstern published the first monograph [38] on the topic of game theory and thus paved the way for its establishment as a discipline in its own right. This work provided a formalisation of utility theory, presented the concepts of the extensive form of a game, the notion of strategy, the assumption of player rationality as well as the concept of mixed strategies. The book also provided examples of the use of game theory in an economic context which no doubt helped integrate the use of game theory into the study of economics.

In order to give an example of a game, Al Tucker presented the tale of the prisoner's dilemma to psychology students in 1950 [17]. This game, which pertains to the problem of cooperation, has since been studied extensively in a variety of applications. One of Tucker's students, John Forbes Nash Jr., proposed the notion of a Nash equilibrium and did fundamental work non-cooperative games [18]. The Nash equilibrium describes a situation in the allocation of strategies to players in which, altering one of the players' strategies, cannot yield an improvement of that player's pay-off value obtained. It is thus difficult to escape such a situation as to each player his or her current choice would seem to yield the best result possible, even if there exists some other distribution of strategies which increases the pay-off value of all players in the game.

3.2 The prisoner's dilemma and the snowdrift game

The prisoner's dilemma has been enormously popular in the study of game theory while the snowdrift game has garnered fair attention in the study of evolutionary biology [5]. The tale of the prisoner's dilemma may be told as follows: Two crooks have been caught by the authorities and are being held in separate rooms. The evidence is somewhat lacking and only suffices for a lesser charge than what is known to have been committed. The authorities, therefore, present each of the hoodlums with an identical offer. Should any prisoner confess to the crime while her partner does not, that prisoner is free to go while the silent accomplice receives a hefty sentence. Should both confess, then they split the prison term equally, while if neither confesses they are only convicted on the minor charge (*i.e.* receive a short sentence) for which the authorities have sufficient evidence [36]. The dilemma arises because no matter what action the accomplice chooses, it is always best to confess. This means that two rational players will both confess and receive the second worst pay-off value in the game, while if both had remained silent, they would have received the second highest pay-off. The game, represented in so-called strategic form, is given by the matrix

$$\boldsymbol{M} = \begin{array}{c c} & \begin{array}{cc} \scriptstyle C & \scriptstyle D \end{array} \\ \begin{array}{c} \scriptstyle C \\ \scriptstyle D \end{array} & \begin{bmatrix} R & S \\ T & P \end{bmatrix} \end{array} \tag{3.1}$$

where T is the temptation to defect, R is the reward for mutual cooperation, P is the punishment for mutual defection and S is the sucker's pay-off. The matrix entry M_{ij} is the pay-off value obtained by the row player playing the strategy in row i against a column player playing the strategy in column j. The game is symmetric and therefore the identity of the players is

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of no consequence to pay-off values obtained — only the choices of strategies are important and so either player can be considered the row player in order to determine his or her pay-off value obtained. In the prisoner's dilemma, these pay-off parameters are assumed to satisfy the inequality chain $T > R > P > S$. Games in this format are called 2×2 games as there are two players with two strategies each.

The game studied in this thesis is the snowdrift game which, in strategic form, is represented by the same matrix as in (3.1), but with the parameters satisfying a different inequality chain, namely $T > R > S > P$. The snowdrift game is attributed to Maynard Smith [16], and is often also called the hawk-dove game. This game may be described in the form of a parable of two motorists held up in a snowdrift. In order for them to continue driving to their destination, snow must first be shoveled. If both shovel, each motorist only shovels half of the snow, and both may continue on their way. If, however, only one shovels, she has to shovel all of the snow, yet both motorists still can continue on their way. If neither shovel snow, they are both stuck and cannot continue on their journey. The extensive form representation of the snowdrift game is shown in Figure 3.1.

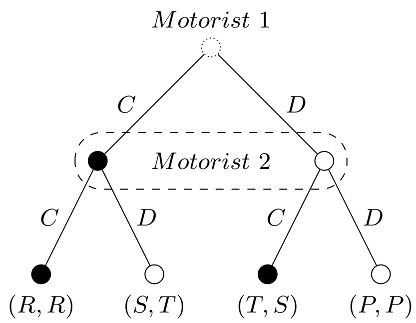

FIGURE 3.1: *Extensive form of the snowdrift game. Each level denotes a player and the edges from a level denote the decisions they can make.*

Extensive form representations of 2×2 games are rooted binary trees, in which each player is represented by a level of the tree and their decisions are represented by the edges of the tree. The outcomes of the game are denoted by pairs of pay-off values payable to the first and second player, respectively. The dotted curve around the two nodes representing the second motorist in Figure 3.1 indicates that those nodes form an information set *i.e.* motorist number two does not know in which of the two states she will find herself. The extensive form of the snowdrift game, in conjunction with the inequality chain of the pay-off values, shows that the optimal strategy in response to an opponent's strategy is the opposite strategy to that of the opponent. Choices are, however, made simultaneously and so the optimal strategy is not clear from the onset of the game.

3.3 Iterated games

Iterated games have been studied as far back as the late 1950s and have proven very influential in investigations related to the notion of altruism (cooperation while defection is more lucrative) and other such phenomena [6]. In 1965, Anatol Rapoport and Albert Chammah published a book on a repeated form of the prisoner's dilemma [28]. Their take on the dilemma was that its interest derives from a partial agreement of outcomes and a partial disagreement, which gives rise to an internal conflict as well as an external conflict. Their study consisted of humans

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playing the prisoner's dilemma repeatedly and analysing the results, both mathematically and psychologically.

In 1971, Robert Trivers highlighted a connection between the repeated prisoner's dilemma and altruistic behaviour in nature. He published an article titled *The evolution of reciprocal altruism* [34] which provided important applications of game theoretic studies in biology as well as a new "solution" to the dilemma. At the time the standard explanation for altruism was kinship, which attributed that kind behaviour among players to a blood relationship between them. Trivers provided an alternative explanation in the form of reciprocity. The article explains that altruism can persevere in instances where the gain is larger than the cost incurred, thus allowing for a win-win situation. This occurs if everyone has a chance to be the receiver and the performer of the altruistic act, in which case the net gain to each individual is positive. This article was the first successful connection between game theory and biology, an intersection which has since become a field of research in its own right.

Iterated versions of 2×2 games yield many interesting results as players can now adopt very intricate strategies. In an iterated game, the players each has recall of the last n games played. Depending on the structure of the game, the value of n may be varied but often consists either of only the previous game, or the entire game history. One important example of such iterated games, in which the players had access to the entire game history, made its appearance during the 1980s. Robert Axelrod hosted two tournaments of the iterated prisoner's dilemma, the results of which he published in a book titled *The evolution of cooperation* [2]. The iterated prisoner's dilemma is the same as the regular prisoner's dilemma with the extension that the players repeat the game a number of times and that the outcome of the game for each player is the summed pay-off values received by that player over all of the games played. Game theorists were invited to submit computer coded programs to play against one another, thus avoiding any human error. The player strategies could be as simple or as complex as the participants wished. The first version involved pitting 15 strategies against one another over a duration of 200 games and, surprisingly, the simplest of all the strategies, called the "tit-for-tat" strategy, won the tournament. The "tit-for-tat" strategy consists of a cooperation during the first game while copying the opponent's previous strategy during all subsequent games. These results were made public and a second round of the competition was opened to anyone who could program a strategy. This time 64 strategies were entered and played against one another. The end point was not predetermined as having played 200 games, but rather occurred stochastically (each game had a probability of roughly 0.3% of being the last). Once again the "tit-for-tat" strategy won the tournament. Furthermore, the results of the tournament showed that altruistic behaviour paid off. This sparked considerable interest in iterated games as well as interest in the study of altruistic behaviour in nature.

3.4 Spatial games

A spatial extension to game theory was pioneered by Martin Nowak and Robert May in two highly influential papers [21, 22] on 2×2 games (with a particular focus on the prisoner's dilemma) played on 200×200 grids in which each player plays the game against his or her nearest neighbours and tally a total score. This score is then compared to the scores of the neighbouring players who compete for that position in the grid, the winner is the player obtaining the highest score during the previous round and places an offspring (a copy of itself) in that position of the grid.

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These papers [21, 22] dealt with two populations of players, playing the strategies of respectively always cooperating or always defecting. This game can be interpreted as a type of cellular automaton exhibiting many simple update rules and interaction across 25 cells in each calculation which is computationally expensive and so its formalisation as a cellular automaton is not preferred. Computer simulations were run instead to determine the effect of spatial structure on the possibility of emergent cooperation over time. The results were both beautiful and astonishing. The motivation for their work was that in the population-based approach adopted in evolutionary game theory, players are assumed to interact with each other player with equal probability, while in the real world, the network in which players find themselves, or the topology of the environment dictates which interactions take place. The addition of spatial structure allows for cooperation to coexist over time in an environment of defectors, provided that these cooperators remain in spatial clusters or groups.

Hauert [9] studied 2×2 games played on a lattice structure using a general formalisation of the pay-off matrix which accommodated a variety of games, such as the prisoner's dilemma, snowdrift and stag-hunt[1] games to name a few, merely by adjusting parameter values and their relationships. The pay-off matrix took the form

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \left[\begin{array}{cc} 1 & S \\ T & 0 \end{array}\right]. \end{array} \tag{3.2}$$

An investigation into mean-field games was conducted (these are populations in which each individual is equally likely to interact with every other individual), and it was found that for the parameter region $S > 0$ and $T > 1$ (which includes the snowdrift game), an equilibrium certainly would be reached which contains both strategies. This equilibrium has a frequency of cooperators of $\frac{S}{S+T+1}$. Furthermore, an investigation into spatial games exhibiting a variety of update rules and involving synchronous as well as asynchronous updating was conducted by means of computer simulation. Some of the findings included that, depending on the relationship between the pay-off parameters, the spatial structure either promotes or inhibits the survival of the strategy of cooperation. Furthermore, the equilibrium states of the game are general and only depend slightly on the initial distribution of strategies, and that higher stochasticity in the spatial game leads to results more similar to those observed in the mean-field game. Specifically for the snowdrift game it was found that the spatial extension inhibits the evolution over time of the strategy of cooperation and that the outcomes of this game are strongly linked to the update rule as well as the initial distribution of strategies (as opposed to other games in which the initial distribution played only a small role).

Since the aforementioned seminal work, many further studies have been conducted in the field of spatial game theory. For a review of spatial games involving coevolutionary rules (rules by which more than the players strategies evolve), the reader may consult [27]. The focus of the current discussion now turns to graphical games. A spatial grid can be represented as a graph and so the distinction between games played on grids and those played on graphs becomes blurred. For the purposes of categorisation, games on grids or lattice structures are considered spatial, while games on other graphs are considered graphical. A natural intersection between these two types of games is games played on cycles, because cycles may be considered one-dimensional lattices. Examples of lattice structures can be seen in Figure 3.2.

Eshel *et al.* [7] studied the drivers of altruistic behaviour in games played on an infinite path as the underlying graph with various radii of interaction and learning among the players. The

[1] Two hunters on the prowl have the options to cooperate to hunt a stag together, or to defect by shooting a hare that crosses their path, therewith scaring away other wild life. The resulting game may be represented by the pay-off matrix (3.1) for which the inequality chain $R > T > P > S$ holds.

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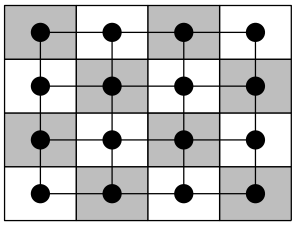
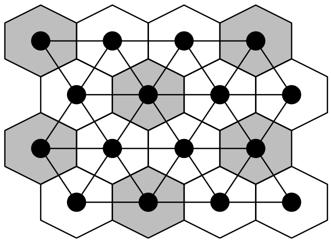


(a) *Square grid and lattice*



(b) *Hexagonal grid and lattice*

FIGURE 3.2: *Square and hexagonal lattice structures superimposed on their respective tilings.*

simple learning assumption of learning by copying successful individuals was made. The purpose of the study was to search for unbeatable strategies (*i.e.* strategies played by the entire population, which cannot successfully be invaded by mutant strategies). The investigation of the game dynamics only involved two strategies at a time, but considered a variety of possible strategies from which to chose these two. It was found that unbeatable strategies were ones that take the welfare of neighbouring players into account. Players along the path play the game against the $2k$ closest players (k players on each side of their own position) in their interaction neighbourhood. Independently, players are given the opportunity to update their strategies (asynchronous updating). Players do not attempt to "learn" a new strategy unless at least one of their direct neighbours played a strategy different to their own. This condition triggers each player to investigate within a learning neighbourhood of $2n+1$ players (n players on either side as well as the player itself) in order to change to a new strategy with a probability proportional to the relative success of the player strategies in the learning neighbourhood. The requirement for a direct neighbour to be playing the opposite strategy implies that only players on the frontier between two strategies can change their own strategies. The investigation began under the assumption that there is only one such frontier at position i, with all players in positions $x \leq i$ playing one strategy and all players in positions $x > i$ playing the other strategy, as depicted graphically in Figure 3.3.

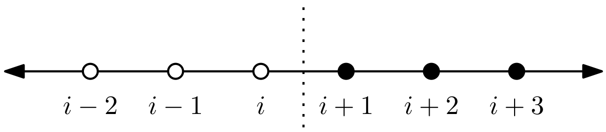


FIGURE 3.3: *The frontier between two clusters of opposite strategies (denoted by the differing vertex colours) on an infinite path as studied by Eshel et al. [7].*

Thereafter, cases were considered in which there are two frontiers, implying a cluster of mutants among a population of individuals playing the same strategy. The game behaviour is then analysable as a random walk of the position of the frontier. The analysis showed that if one strategy α has frontier advantage over another strategy β, denoted by $\alpha \succ \beta$, then the entire population will eventually play strategy α. Frontier advantage is determined by a comparison between the probability of a shift in the frontier position toward either side. Furthermore, strategy α is unbeatable if and only if strategy α has frontier advantage over any chosen strategy β. The results showed in particular that $\alpha \succ \beta$ if

$$\begin{aligned}
&\big((2k+n-1)\phi(\alpha,\alpha)+(2k-n+1)\phi(\alpha,\beta)\big)\big((2k+n)\phi(\alpha,\alpha)+(2k-n)\phi(\alpha,\beta)\big)\\
&\quad > \big((2k+n-1)\phi(\beta,\beta)+(2k-n+1)\phi(\beta,\alpha)\big)\big((2k+n)\phi(\beta,\beta)+(2k-n)\phi(\beta,\alpha)\big),
\end{aligned}$$

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for $n \leq k$, where $\phi(x, y)$ is the pay-off value obtained by a player adopting strategy x against a player adopting strategy y. Similarly, $\alpha \succ \beta$ if

$$\begin{aligned}&\big((4n-k-1)\phi(\alpha,\alpha)+(k+1)\phi(\alpha,\beta)\big)\big((4n-k+3)\phi(\alpha,\alpha)+(k+1)\phi(\alpha,\beta)\big)\\&\quad>\big((4n-k-1)\phi(\beta,\beta)+(k+1)\phi(\beta,\alpha)\big)\big((4n-k+3)\phi(\beta,\beta)+(k+1)\phi(\beta,\alpha)\big)\end{aligned}$$

for $n \geq k$.

Conditions were derived for unbeatable strategies for large values of k and n, keeping the ratio $\frac{n}{k}$ constant. The main results involved comparing the outcome of the original version Π of a game investigated, and that of another game Π^r in which the pay-off matrix is adjusted to account for a relationship between players, thus incorporating kinship, where r is a measure of relatedness. An important finding was that unbeatable strategies in Π are necessarily also unbeatable in Π^r, which indicates the importance of considering the welfare of others alongside the welfare of oneself in the determination of unbeatable strategies.

3.5 Graphical games

Spatial games are a special class of graphical games in which the underlying graph is a lattice, which can be square, hexagonal, or one-dimensional. In graphical games, the underlying graph is, however, not restricted in this way; the game can in fact be played on any type of graph. Playing a game on a graph involves specification of an *interaction graph* which determines the games that are played between pairs of players and a *replacement graph* that determines each players region of influence (learning neighbourhood). In some cases the interaction graph and the replacement graph are different [7, 25] while in others they are the same. A large body of work also exists in the realm of graphical games otherwise known as evolutionary game theory. Graphs are constructed in a variety of forms (amongst others, regular graphs, random graphs, small-world networks, and scale-free networks) and the evolution of the strategy of cooperation is studied.

Lieberman *et al.* [14] studied the effects of graph structures in this context on the notion of a so-called fixation probability. The fixation probability is the likelihood that the strategy of a singular mutant introduced into a graph will dominate the population of strategies during later generations. The process of natural selection, which is modelled to some degree by these games, determines that the fitness of an individual (player in the population) determines the likelihood of that player passing on its "genetic material" (its adopted strategy) to future generations of players. The strength of selection denotes the importance of a player's performance in the game when determining its fitness. Strong selection occurs when the player performance is taken directly as the player fitness, while weak selection involves keeping fitness relatively similar across all players, but with a small perturbation provided by each player's performance during the game. Given a weighted graph with a homogeneous population (all players adopting the same strategy) occupying the vertices of the graph, a player is randomly selected for mutation, therewith changing its strategy to the mutant strategy. The relative fitness of this mutant player is denoted by r. Thereafter, an individual is chosen to replicate according to its fitness. A neighbour of the chosen individual is selected to be replaced by a probability relative to the weight of the edge connection between the two. Lieberman *et al.* [14] found that in the scenario of constant selection (where the fitness of an entity is independent of the strategy proportions of the population), the underlying graph can tip the scales toward random drift or selection. They also found that in the case of selection depending on frequency, the underlying graph can change the nature of the selection process completely, favouring even dominated species. Their work

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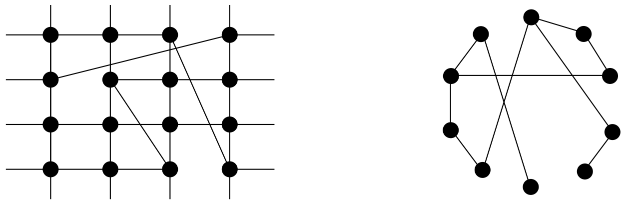
showed that the study of the networks of such interactions play an important role in determining the state of the entire network.

Roca *et al.* [29] investigated evolutionary games on regular lattice structures, random networks, and small-world networks (all with degree homogeneity). Small-world networks are regular graphs to which some reconnecting of edges has taken place. Figure 3.4 contains two graphical examples of small-world networks. The game investigated by Roca *et al.* was a versatile representation of a variety of 2×2 games, namely the prisoner's dilemma, the snowdrift game, the stag-hunt game as well as the harmony game[2]. The pay-off matrix used was identical to that of Hauert [9] in (3.2), with parameter values $S \in (-1, 1)$ and $T \in (0, 2)$. The relationship between the parameters adopted, determine which game is being investigated. The investigation was aimed at deciphering whether the promotion of the cooperation strategy was a result of the spatial aspect, the update rule or perhaps the coupling of the two together.



(a) (b)

FIGURE 3.4: *Two examples of small-world networks (a) with and (b) without degree homogeneity.*

Some of the results obtained by Roca *et al.* [29] pertaining to 2×2 games are briefly summarised here. In the case of synchronous updating, they found that indeed cooperation in the snowdrift game was inhibited in the setting of stochastic update rules, which is corroborated by the work previously conducted by Hauert and Doebeli [10]. Their investigation showed that the clustering present in the network played a significant role in determining the influence of the network structure on the evolution of cooperation, with higher clustering resulting in greater differences between regular lattice arrangements and random networks with the same regular degree. The difference is that when there is a low degree of clustering, the effect of the network structure is essentially only that of including repeated interactions with a few interactions per round, while when the clustering coefficient is larger, then clusters can form in which groups of three players all play the game against one another. This led their investigation to seek to determine whether this property of clustering was robust and so an investigation into small-world networks (similar to the graph shown in Figure 3.4(a)) followed, in which regular lattices were constructed with some reconnecting of edges, so that the large clustering property remained intact while dramatically decreasing the diameter of the network (the longest shortest path between any two vertices).

Another conclusion drawn was that weak selection further weakens the effect of the spatial structure and moves the result of the evolutionary process closer to what is found in well-mixed populations. The positive effect of the spatial structure is most robust in terms of different update rules in the stag-hunt game, while the effect of stochastic rules inhibit cooperation in the snowdrift game. Roca *et al.* explained this opposite effect in the two games by comparing the structured version of the game with the spatial game in terms of the global density of cooperators and the local density of cooperators. The equilibrium local density of cooperators is reached sooner and thus less cooperation is required for the equilibrium distribution of strategies in

[2] A 2×2 game adopting the pay-off matrix (3.1) following inequality chain $R > T > S > P$.

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the snowdrift game, while the opposite is true for the stag-hunt game. It is important to note that these effects prevail in the context of fairly high clustering in the network. The effect of clustering is explained by the presence of random groups of cooperators influencing their defecting neighbours to become cooperators, whose other neighbours (because of clustering) are likely also to be cooperators. Locality results in players whose visible networks consist largely of cooperators which, in turn, allows for the spread of the strategy of cooperation in the stag hunt game.

3.6 Evolutionary games on cycles

Ohtsuki and Nowak [24] undertook an analytical investigation of evolutionary games on cycles. Their work was aimed at finding an exact expression for the fixation probabilities in both the cases where the interaction and the replacement graphs were the same cycle graph and where the interaction graph is complete while the replacement graph is a cycle. The results for the first of these cases are summarised here. Three of the most popular update rules were investigated, namely *birth-death*, *death-birth*, and *imitation*. The first of these describes the scenario in which a player is chosen to be duplicated proportionally to its fitness and subsequently a neighbour is replaced by this duplicate so that both positions adopt the strategy of the duplicated individual. The second update rule, death-birth, dictates that a player is chosen to die out at random after which its position is competed for by the nearest neighbouring players. The last rule, imitation, dictates that a player is chosen at random and its strategy is updated to that of one of its neighbours or else retained, proportionally to the player's relative fitness in comparison with that of its neighbours. An homogeneous population of cooperators (defectors) was considered to which a singular mutant defector (cooperator) was added. The game then played out in terms of the selection and update protocols. The quest was to determine the fixation probabilities of the strategies of cooperation and defection (*i.e.* the probability that the mutant strategy is adopted by the entire population in both these cases). When comparing the fixation probabilities of the two strategies, a conclusion may be drawn in terms of which strategy is favoured in the context investigated.

Ohtsuki and Nowak [24] proceeded to derive conditions for the favouring of one strategy over the other in the context of 2×2 games in terms of the pay-off parameters for both weak and strong natural selection and in the context of each of the three update rules described. The result specifically for strong selection and an imitation update rule (but general to 2×2 games represented by (3.1)) is that the strategy of cooperation is favoured if $(3R + S)(R + S) > (T + P)(T + 3P)$. Their investigation extended to the pay-off matrix

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \left[\begin{array}{cc} b-c & -c \\ c & 0 \end{array}\right], \end{array} \tag{3.3}$$

a popular normalisation of the pay-off parameters in 2×2 games. The results obtained (for the prisoner's dilemma in the form (3.3)) in the context of the imitation update rule and weak selection are that cooperation is favoured when $\frac{b}{c} > 4 + \frac{18}{N-6}$, where N is the order of the underlying cycle, b is the benefit provided by playing against a cooperator and c is the cost incurred by a cooperator.

An investigation by Burger *et al.* [4] during the period 2012–2013 took quite a different shape than what has been discussed thus far. They adopted a deterministic approach toward the evolutionary prisoner's dilemma played on a cycle. Strong and global selection was implemented.

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This means that during each round, every player compares its pay-off value obtained with those of its neighbours along the cycle and subsequently changes its strategy to that of the highest-scoring player in its neighbourhood. This includes the possibility of synchronous updating (the case where one player imitates another whose strategy too may have changed during the following round). The investigation was not concerned with fixation probabilities, but rather the probability of and conditions for persistent cooperation, given the order of the underlying cycle and assigning initial strategies at random to the players in the game. Furthermore, a characterisation of the steady states of the game was undertaken which culminated in an enumeration of these. As has been mentioned, strong selection is an avenue of interest which has not received much attention due to its divergence from neutral drift [1]. Furthermore, synchronous updating also adds a dimension of reality [30].

Laird [12], considered pay-off matrices in which ties between cooperators and defectors is possible in the evolutionary snowdrift game played on cycles. Such pay-off parameters allow for the interactions required for *standoffs*, which are distributions of strategies amongst the players in the graph in which both cooperation and defection coexist, but no longer expand or decline. In the game investigated by Laird, a random player and its neighbour are chosen and, if the neighbour of the chosen player achieves a larger mean pay-off value, the chosen player changes its strategy to that of the neighbour by a probability proportional to the difference between their mean pay-off values. Laird found that standoffs may occur at certain equalities of the pay-off parameter values, namely when $T = 1 + S$ and when $T = 2S$. In order for the first of these equalities to result in a standoff, clusters of cooperators and defectors and an absence of singletons is required. In order for the second of these equalities to result in a standoff, cooperators are required to be dispersed as singletons between clusters of defectors. These findings are important as a similar analysis is conducted later in this thesis, but with a different update rule (synchronous updating) as well as a focus instead on the regions between these equalities (*i.e.* regions of inequality).

3.7 Chapter summary

This chapter was devoted to a review of the literature relevant to the topic of this thesis with special interest in developments in and toward spatial evolutionary game theory. A brief overview of classical game theory was provided in §3.1. This was extended in §3.2 to an elucidation of two well-known archetypal games known as the prisoner's dilemma and the snowdrift game. Evolutionary games and their muse, iterated games, were discussed in §3.3. It was observed how the development of iterated games led to the study of the emergence of altruism in social dilemma games. One of the answers to the resolution of these dilemmas came in the form of spatial games played by populations on lattice structures, as discussed in §3.4. Games in which the population structure can be determined by any graph (not only lattices) are known as graphical games. Some of the seminal work in graphical games was highlighted in §3.5. Finally, work on the special class of evolutionary games on cycles was reviewed in §3.6.

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CHAPTER 4

Representing the game

Contents

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This chapter is devoted to a description of various notions required to represent the ESS and to analyse the asymptotic long-term strategies of its players. The chapter opens in §4.1 with a discussion on how the ESS works in terms of player strategy determination over a number of game rounds, based on a pay-off matrix. The context of analysis in this thesis is elucidated in §4.2 by scoping the underlying structure according to which pairs of ESS players compete against each other to include cycles only. A normalisation procedure is carried out in §4.3 in order to reduce the number of parameters in traditional descriptions of the game to a number that renders the subsequent analysis more manageable. This is followed by a discussion in §4.4 on a sate graph representation of the ESS game progression over its various rounds and it is shown that the long-term asymptotic strategies of players depend on various relations between the game parameters, captured succinctly in the form of a parameter phase plane. The concept of a fixation probability is next introduced in §4.5. The chapter is brought to a close in §4.6 by presenting a brief summary of its contents.

4.1 The game pay-off matrix

The matrix form of a two-person game contains the various pay-off values of players in matrix form, payable to the row player. Consider a row player having chosen to play strategy i and that its opponent, the column player, has chosen strategy j. Then the pay-off due to the row player may be found in row i and column j of the pay-off the matrix. This is also the case in evolutionary game theory, although, it is important also to include a representation of the underlying structure of the game. Various pairs of players are set to play against one another in evolutionary game theory. This is conveniently represented by a graph, called the underlying graph of the game in which the vertices represent the players and their strategies while edges of the graph join vertex representations of players that are set to play against one another. In

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the context of the ESS considered in this thesis, games are represented by the pair $\Upsilon = (\Pi, G)$, where

$$\Pi = \begin{array}{c c} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{bmatrix} R & S \\ T & P \end{bmatrix} \end{array}, \tag{4.1}$$

is the matrix of pay-off values (of the row player), and G is the underlying graph of the game.

In the Snowdrift game, much like in the Prisoner's Dilemma, players chose to defect (D) or to cooperate (C). This choice (and those of their opponents) then results in one of the four pay-off values being awarded to each of the members of each player pair. These values are T, the temptation to defect, R, the reward for mutual cooperation, P the punishment for mutual defection and S the suckers pay-off (awarded to a cooperating player who faces defection by its partner/opponent). In the matrix form of the game, the rows represent the strategies of the row player, the columns represent the strategies of the opponent (the column player) and the matrix entries are the pay-off values obtainable by the row player for the combinations of strategies. These parameters are assumed to satisfy the inequality chain $P < S < R < T$.

Suppose the vertices of the underlying graph G for the ESS are labelled $1, \ldots, n$. Then a *state* of the ESS during any particular round of the game is denoted by a binary string $\boldsymbol{s} = s_1, \ldots, s_n$, where $s_i \in \{C, D\}$ denotes the strategy adopted by the player represented by vertex $i \in \{1, \ldots, n\}$ of G. A *cooperation run* (*defection run* respectively) is a maximal contiguous substate of an ESS game state containing cooperators (defectors, respectively) only. A cooperation run of length $i \geq 3$ is abbreviated as $\langle C \rangle^i$, while a defection run of length $i \geq 3$ is abbreviated as $\langle D \rangle^i$. Any state of the ESS on G may be represented graphically as a bi-colouring of the vertices of G in which vertices of one colour represents cooperators and vertices of the other colour represent defectors. In this thesis, the convention is followed throughout that cooperators are denoted by solid vertices and defectors by open vertices in any such graphical representation of a game state.

While a labelled underlying graph G is required to encode the state of the ESS as a binary string, the emergence of substructures of cooperation and defection among players strategies from the game state during any particular round of the ESS to another state during the next round, of course, does not depend on the particular vertex labelling adopted. In particular, two game states $\boldsymbol{s} = s_1, \ldots, s_n$ and $\boldsymbol{s}' = s'_1, \ldots, s'_n$ of the ESS are *automorphic* if there exists an automorphism $f : \{1, \ldots, n\} \mapsto \{1, \ldots, n\}$ such that (a) two vertices i and j are in adjacent in G if and only if the vertices $f(i)$ and $f(j)$ are adjacent in G, for all $i, j \in \{1, \ldots, n\}$ and (b) $s_{f(i)} = s'_i$ for all $i \in \{1, \ldots, n\}$ (that is, the function f is a relabelling of the vertices of G that preserves player adjacency in G as well as player strategies). An *automorphism class* of ESS states is a maximal set of game sates such that any two states in the set are automorphic to one another. The *class leader* of such an automorphism class is the lexicographically first member of the class (taking $C < D$). Finally, the *weight* of an ESS automorphism class is the number of players adopting the strategy of cooperation within any member of that class.

For example, the $2^5 = 32$ distinct states of the ESS on the "house graph" of order 5, shown in Figure 4.1, may be organised into twenty automorphism classes, also shown in the figure. The class leader of each class is shown in black and white, while other members of a class are shown in grey scale. The weights of the automorphism classes are also indicated in the figure.

The rule according to which an ESS player updates its strategy during each game round is simply that the player adopts the strategy during the next round of the player within its neighbourhood which achieves the largest normalised pay-off value (the total pay-off value received from game opponents in its open neighbourhood, divided by the cardinality of that open neighbourhood)

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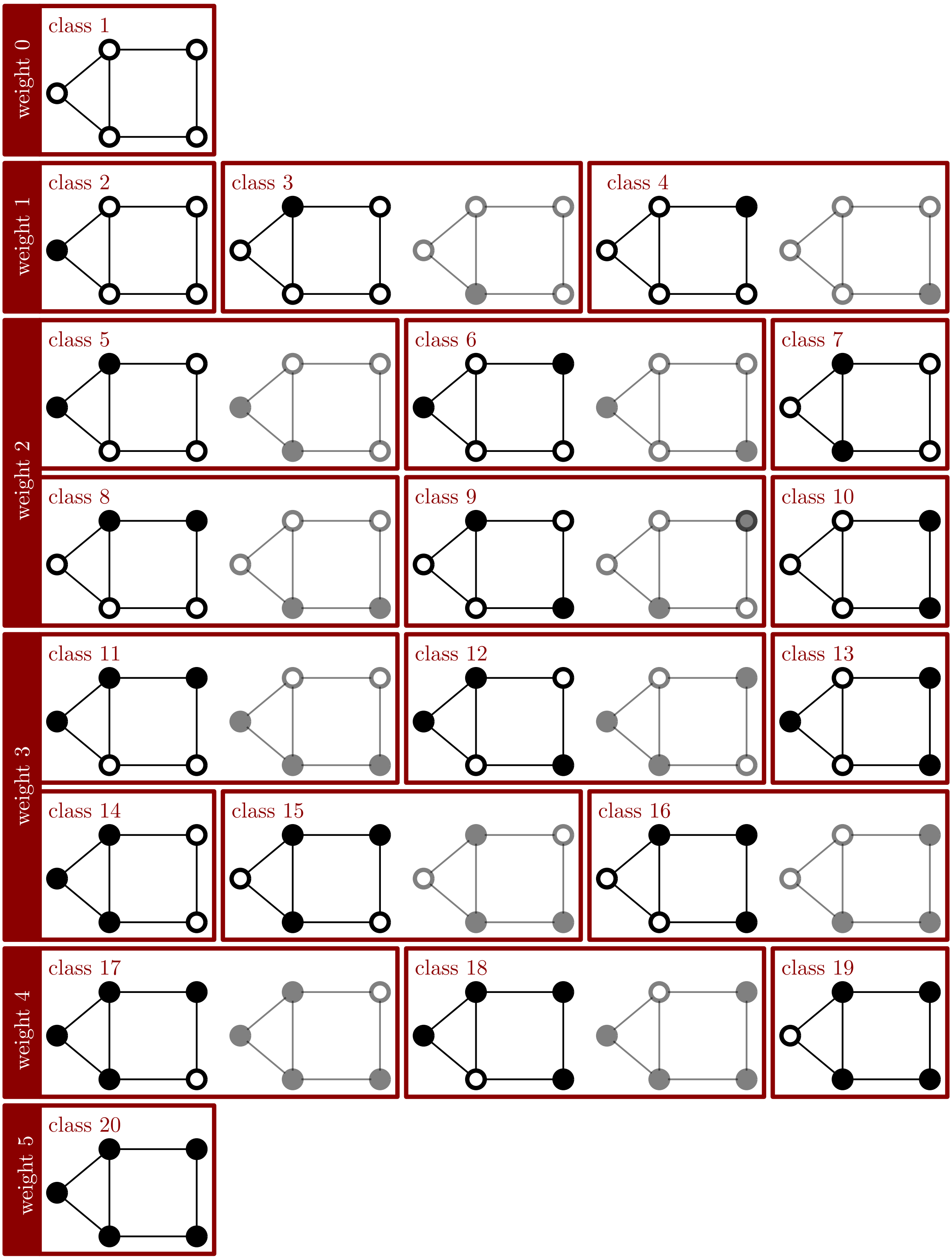

FIGURE 4.1: *The thirty two game states of the ESS played on the "house graph," partitioned into twenty automorphism classes. The strategy of cooperation is represented by a solid vertex while that of defection is represented by an open vertex. The class leader of an automorphism class of game states is shown in black and white, while other state members of the class are shown in grey scale.*

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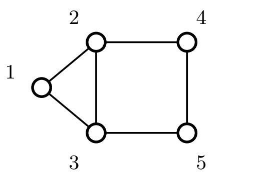
during the current round. In the case of a tie between the normalised pay-off values received by two neighbouring players adopting opposite strategies, the player under consideration retains its own strategy. The normalised pay-off value of each player is henceforth called its *score*. The progression of strategies thus played during an ESS instance may be illustrated graphically in the form of a series of graphs, one for each successive game round, in which player strategies are indicated by means of bi-colourings of the vertices of the underlying graph, as described above.

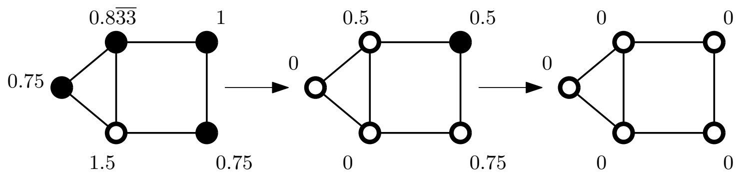
Adopting the vertex labelling of the "house graph" shown in Figure 4.2(a) for example, the first ESS state in Figure 4.2(b) is $CCDCC$. This state is the class leader of the fourth automorphism class in Figure 4.1 and is automorphic to the state $CD\langle C\rangle^3$. If the pay-off matrix of the ESS instance is

$$\mathbf{\Pi} = \begin{array}{c c} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{3}{2} & 0 \end{bmatrix} \end{array},$$

then the latter state transitions to the state $\langle D\rangle^4 C$ in one game round (as shown in the middle graph of Figure 4.2(b). This is because Player 2, being a cooperator, receives pay-offs of 1 from Player 1 (also a cooperator), 1 from Player 4 (also a cooperator), and $\frac{1}{2}$ from Player 3 (a defector) during the first game round. The score of Player 2 is therefore $\frac{5}{6} = 0.8\overline{33}$, as shown in the figure. The score of Player 3 is similarly $\frac{3}{2} = 1.5$. Since the score of Player 3 (a defector) is larger than that of Player 2 (a cooperator), Player 1 updates its strategy to that of defection during the next round of the game. Similar arguments may be followed to verify that Players 2 and 4 also change their strategy to defection during the next game round, as indicated in the middle graph of Figure 4.2(b). According to similar computations, the resulting state $\langle D\rangle^4 C$ ultimately transitions to the all-defector state $\langle D\rangle^5$ within a further game round (as shown in the last graph of Figure 4.2(b)). Thereafter, the game state remains $\langle D\rangle^5$ during all subsequent rounds.



(a) *Labelling of vertices*



(b) *An example of a game progression of ESS rounds*

FIGURE 4.2: *An example of the progression of the game on the "house graph".*

4.2 The underlying graph

The scope of analysis of the ESS is limited to the rather benign, yet not boring, class of cycle graphs in this thesis. Recall that cycle graphs are 2-regular, connected graphs (*i.e.* each vertex has degree 2). The reason why cycles are considered is that they are a natural simplification of the grid graph on which the seminal investigations by Nowak an May [21, 22] were based. In Figure 4.3, two equivalent graphical representations of an ESS state on a cycle graph of order 6 are illustrated. The player strategies in this particular pair of graphical representations are captured by the binary string $CCDCDC$, which is automorphic to $CCCDCD$, the class leader of that set of automorphic game states. While the representation in part (a) is intuitive, it can be simplified for the sake of brevity to the graphical representation in part (b) of the figure, in which the dashed edge will be omitted in future as its indication of wrapping around will be clear from the context.

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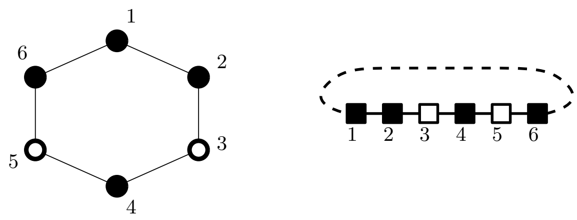

(a) (b)

FIGURE 4.3: *Graphical representations of the strategies of six players making up the game state $CCDCDC$ in the ESS played on a 6-cycle as underlying graph.*

4.3 Normalisation of the pay-off matrix of the game

It is beneficial to be able to analyse the ESS game dynamics using fewer than four pay-off parameter values. This may be achieved by setting two of the game parameters (pay-off values) equal to 0 and 1, respectively. This leaves only two parameters that can be varied.

Consider two game instances, played on the same underlying graph G, but with two different sets of pay-off values: $\Pi = \{T, R, S, P\}$ and $\hat{\Pi} = \{\hat{T}, 1, \hat{S}, 0\}$. These game instances are denoted by: $\Upsilon = (\Pi, G)$ and $\hat{\Upsilon} = (\hat{\Pi}, G)$, respectively. In what follows, it is shown that these two game instances can be considered equivalent under a certain pair of conditions.

Denote the highest scoring players in player p's closed neighbourhood playing the strategies C and D by c and d in game instance Υ. Suppose the scores of these two players are ψ_c and ψ_d, respectively, in game instance Υ, and $\hat{\psi}_c$ and $\hat{\psi}_d$, respectively, in the game instance $\hat{\Upsilon}$. The strategy of player p is determined by the difference between the scores of these two players as any player chooses the strategy of its best scoring neighbour of the current round during the next round. Let $\chi(p)$ be the strategy of player p during the current game round. Player p's strategy during the next round is therefore

$$f(\Pi, \chi) = \begin{cases} C & \text{if } \psi_c > \psi_d \\ D & \text{if } \psi_d > \psi_c \\ \chi(p) & \text{if } \psi_d = \psi_c. \end{cases}$$

Now let x be the proportion of players in the open neighbourhood of c playing the strategy C and similarly let y be the proportion of players in the open neighbourhood of d playing the strategy C. If $\psi_d > \psi_c$, then

$$xR + (1-x)S < yT + (1-y)P. \tag{4.2}$$

Subtracting P from each side of the inequality in (4.2) yields

$$xR + (1-x)S - P < yT + (1-y)P - P, \tag{4.3}$$

which can be rewritten as

$$xR + S - xS - P < yT + P - yP - P. \tag{4.4}$$

Factorisation of (4.4) therefore yields,

$$x(R-P) + (1-x)(S-P) < y(T-P) \tag{4.5}$$

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and, after dividing both sides of (4.5) by $R-P$ (which is positive), it follows that

$$x+(1-x)\frac{S-P}{R-P}<y\frac{T-P}{R-P}. \tag{4.6}$$

Now in order to render the game instances Υ and $\hat{\Upsilon}$ equivalent, it is required that

$$\hat{T}=\frac{T-P}{R-P} \text{ and } \hat{S}=\frac{S-P}{R-P}, \tag{4.7}$$

which results in

$$x+(1-x)\hat{S}<y\hat{T}. \tag{4.8}$$

This makes it clear that $\psi_d > \psi_c$ implies $\hat{\psi}_d > \hat{\psi}_c$. By similar arguments it can be shown that $\psi_d < \psi_c$ implies $\hat{\psi}_d < \hat{\psi}_c$ and that $\psi_d = \psi_c$ implies $\hat{\psi}_d = \hat{\psi}_c$. The two game instances $\hat{\Upsilon}$ and Υ may therefore be considered equivalent under the parameter transformation in (4.7).

4.4 The state graph and the (S, T)-phase plane

The strategy progression of the ESS from round to round may be described graphically, for all possible initial game states, by means of a so-called *state graph*. This graph is a vertex-labelled directed pseudodigraph in which the vertices represent entire game states and in which an arc (or directed edge) is present from a state $\boldsymbol{s}$ to a state $\boldsymbol{s}'$ if the game progresses from the state $\boldsymbol{s}$ to the state $\boldsymbol{s}'$ within a single round. In this way, the state graph succinctly captures the game dynamics as the ESS progresses from round to round, starting from any initial state. A game state $\boldsymbol{s}$ is said to *attract* another state $\boldsymbol{s}'$ if there exists a (directed) path from $\boldsymbol{s}'$ to $\boldsymbol{s}$ in the state graph. The state graph of the ESS on the "house graph" in Figure 4.2(a) is shown in Figure 4.4. In particular, the directed path representing the progression of game states in Figure 4.2(b) is indicated by a dashed curve in the state graph of Figure 4.4.

The pay-off values possible for players in a cyclic incarnation of the ESS are rather limited, which allows for an intuitive representation of the various regions of pay-off parameters in the (S, T)-phase plane. An investigation into the nature of the ESS leads to the identification of four distinct regions of interest in this plane, as indicated in Figure 4.5. These regions may be found by considering the pay-off values possible in the ESS on a cycle. The pay-off values achievable by a defector are $\{0, T, 2T\}$, while the pay-off values achievable by a cooperator are $\{2S, S+1, 2\}$. The isoclines in the (S, T)-phase plane where these pay-off values coincide demarcate four regions in which the asymptotic long-term player strategies of the game differ from one another.

The result of the following theorem due to Burger *et al.* [3], although intended for the ESPD with pay-off matrix

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{bmatrix} 1 & 0 \\ T & P \end{bmatrix}, \end{array} \tag{4.9}$$

also holds for the case of the ESS.

Theorem 4.1 (Properties of the state graph of the ESPD [3]).
Let G be any connected graph of order n with maximum degree Δ. If $T > \Delta(1-P)+P$, then the state graph of the ESPD on G has two components. Moreover, one of these components comprises the all-cooperator steady state $\langle C\rangle^n$ only, while the other component contains all other game states, including the all-defector steady state $\langle D\rangle^n$ which attracts all states except $\langle C\rangle^n$.

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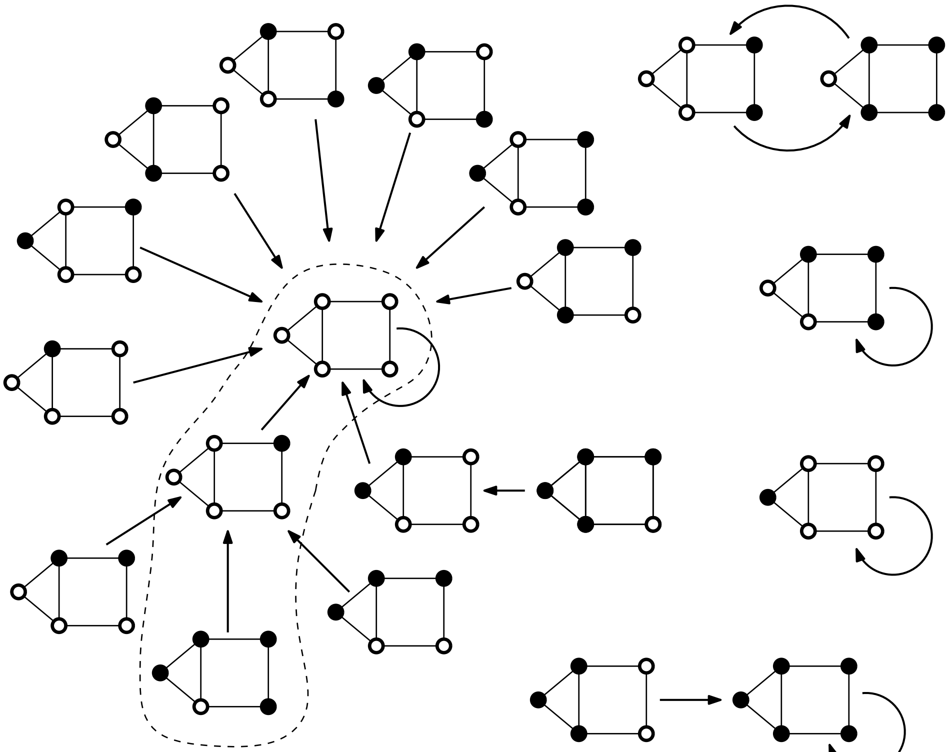
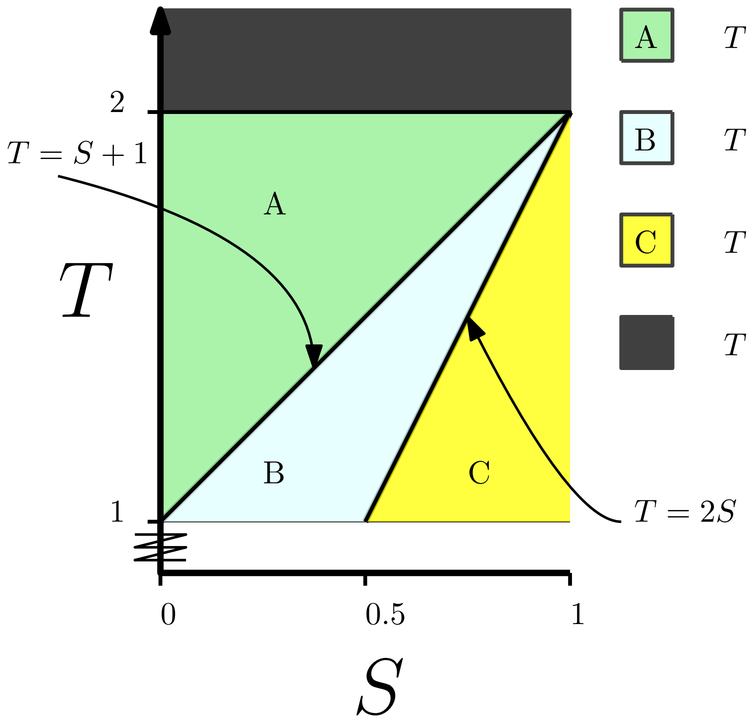


FIGURE 4.4: *The state graph for the ESS with the "house graph" of Figure 4.2(a) as underlying graph and* $\Pi = \{\frac{3}{2}, 1, \frac{1}{2}, 0\}$. *A solid vertex denotes the strategy of cooperation, while an open vertex denotes the strategy of defection. The dashed curve highlights the game progression in Figure 4.2(b).*



A	$T > 2S$,	$T > 1 + S$,	$T < 2$
B	$T > 2S$,	$T < 1 + S$,	$T < 2$
C	$T < 2S$,	$T < 1 + S$,	$T < 2$
	$T > 2S$,	$T > 1 + S$,	$T \geq 2$

FIGURE 4.5: *The (S,T)-phase plane of the ESS on a cycle. Four regions of the (S,T)-phase plane are indicated in the figure. The long-term asymptotic game dynamics differ fundamentally in these regions.*

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It is clear from the context of the ESS as normalised in §4.3 that the pay-off values obtainable are such that a defector's lowest pay-off value achievable, whilst playing against a cooperator in the scenario where $T > \Delta$, is still larger than the highest pay-off value achievable by a cooperator. As stated by Burger *et al.* [4], the result is that the region corresponding to large values of T (*i.e.* the upper-most region in Figure 4.5) can be considered uninteresting. In other words, normalised pay-off parameter values imply that values of $T < 2$ should inherently be more interesting than those where $T \geq 2$ as the dominance of the strategy of defection subsides and the strategy of cooperation is afforded a chance to persist in the former case, but not in the latter case.

In Figure 4.6, the state graphs of the ESS on a 6-cycle may be seen for each of the regions below the line $T = 2$ in the (S, T)-phase plane of Figure 4.5. It is clear from the figure that the game dynamics are fundamentally different in the various regions of the phase plane.

4.5 Fixation probabilities

In population genetics, the notion of a fixation probability is used to examine the relative abundance of alleles. More specifically, it is the probability that an allele of interest attains a 100% level of frequency within the population being studied [26]. Interest in this notion arises because of its usefulness in establishing estimates of probabilities according to which a drug resistance may be developed in a population or the probability of adaptation to environmental features.

In game theory, the concept of an evolutionary stable strategy was proposed by Maynard Smith [16]. An evolutionary stable strategy is a strategy whose expected pay-off value in a game against itself is greater than the expected pay-off value of any other strategy playing against this strategy. Populations of individuals all playing the same evolutionary stable strategy cannot be invaded by players of mutant strategies. Axelrod [2, Chapter 3] explored this notion, affording attention to the possibility of invasion of a new strategy and investigating what proportion of a well-mixed population is required to be playing the mutant strategy for invasion to be possible. The notion of fixation probability has since been established in the study of evolutionary game theory.

Considering a homogeneous population (*i.e.* a population of individuals all playing the same strategy), a mutant strategy is defined as an entering strategy that is different from that of the rest of the population. A player playing this mutant strategy is referred to as a *mutant*. In the context of evolutionary game theory, a fixation probability can be interpreted as the probability of a mutant strategy taking over the strategy of a population into which it is introduced. In the case of the ESS, the question is whether a population of defectors can be taken over by the mutation of a single player to the strategy of cooperation, or *vice-versa*. In cases where a single mutant is incapable of inducing a game state in which any more than one mutant persists, the definition of the fixation probability may be generalised to probability of a group of players undergoing the same mutation in a population and the strategy of these mutants taking over the entire population. These probabilities are often pursued in a stochastic setting, but in the deterministic investigation of this thesis a similar analysis is possible. For the analysis of the ESS on cycles, the fixation probability is defined as the probability that a group players mutating to the opposite strategy to that of the homogeneous population, achieve a game state in which the mutant strategy cannot be eliminated in the future and has exhibited growth since

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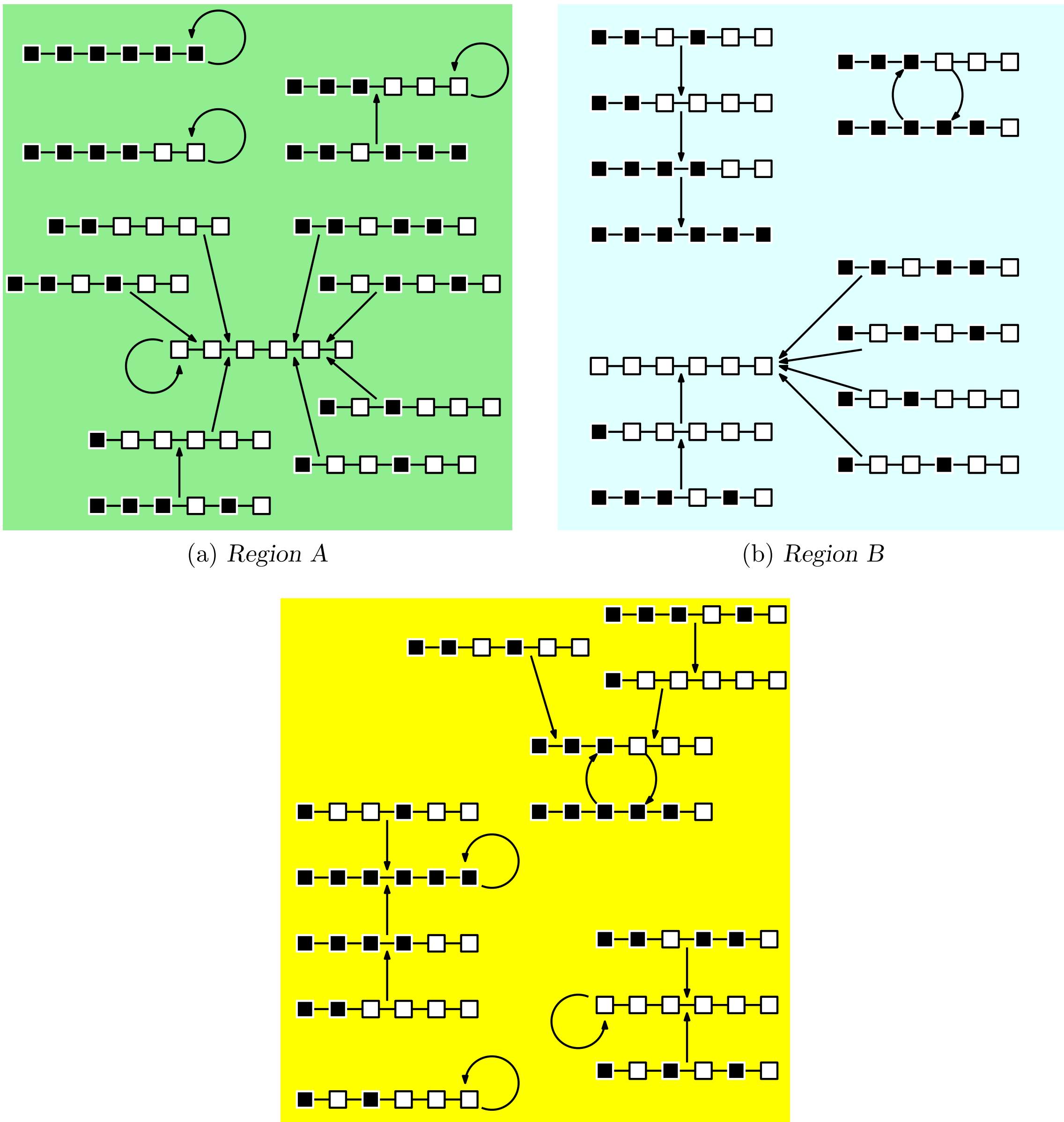

(a) *Region A* (b) *Region B*

(c) *Region C*

FIGURE 4.6: *The state graphs for the ESS on a 6-cycle in the three regions A, B and C of the (S, T)-phase plane in Figure 4.5. A solid square denotes the strategy of cooperation, while an open square denotes the strategy of defection. Although players are represented in linear arrays, these arrays should be interpreted as wrapping around so that the first and last player in the array are adjacent, as illustrated in Figure 4.3.*

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its inception[1]. It is denoted in this thesis by $F(n, k)$, where n is the order of the underlying cycle and $k < n/2$ is the size of the group of players undergoing the initial mutation.

Consider, as an example, the "house graph" in Figure 4.2(a) again, with pay-off parameter values $\Pi = \{T = \frac{3}{2}, R = 1, S = \frac{1}{2}, P = 0\}$. Using Figures 4.1 and 4.4 it is possible to find the fixation probability of an initial group of players mutating to the strategy of defection ranging from size 1 to 2 by inspection. The strategy of defection will fix the population of players as the all-defector state if a singular cooperator mutates to adopt the strategy of defection on any of the vertices 2, 3, 4 or 5. Therefore, the fixation probability of a singular defector on the "house graph" with pay-off parameters as described is $\frac{4}{5}$. For a group of two players initially mutating to adopt the strategy of defection that are chosen in one of ten possible ways ($\binom{5}{2} = 10$), seven of these choices lead to the all-defector steady state, which implies that the fixation probability of the defection strategy is $\frac{7}{10}$. The seven player choices that lead to fixation of the defection strategy are given in Table 4.1.

TABLE 4.1: *Pairs of vertices of the "house graph" that, if chosen for initial mutation to the defection strategy, lead to fixation of the defection strategy.*

Player 1 of the pair:	1	1	2	2	2	3	3
Player 2 of the pair:	4	5	3	4	5	4	5

4.6 Chapter summary

This chapter opened with a discussion on the ESS pay-off matrix in §4.1. A short description of the underlying graph of the ESS considered in this thesis was provided in §4.2. A normalisation of the pay-off parameters of the ESS was conducted in §4.3 which allows for a more manageable manipulation of the pay-off parameters during investigations into the long-term strategy adoption dynamics of the game. The notion of the (S, T)-phase plane, a graphical representation of the normalised pay-off parameters, was introduced in §4.4, along with the notion of a state graph which is used to capture the long-term strategy adoption behaviour of players of the game from all possible initial conditions. The concept of a fixation probability was finally described in §4.5, referring to its origins in genetics and discussing its use in game theory as well as the modified incarnation adopted in this thesis.

[1]Cooperation and defection exhibit fundamentally different behaviour and thus the refinement of the definition of fixation probability is left until this behaviour is elucidated.

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CHAPTER 5

The case where $2S < S + 1 < T$ (Region A)

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A long-time asymptotic analysis is conducted in this chapter of the ESS on a cycle in Region A of the (S, T)-phase plane in Figure 4.5. The chapter opens in §5.1 with a comparison between the ESS and the ESPD on a cycle in the region of the (T, P)-phase plane where $T + P \leq 2$, as investigated by Burger *et al.* [4]. The result of this comparison is an isomorphism between the two evolutionary games played on the cycle graph. The remainder of the chapter is then devoted to a restatement of the results for the ESPD within the context of the ESS with respect to the requirements of persistent cooperation in §5.2, the probability of persistent cooperation in §5.3 and the enumeration of components of the state graph in §5.4. The possibility of fixation of the strategy of cooperation is considered briefly in §5.5, after which the chapter finally closes with a summary of its contents in §5.6.

The results of this chapter hold if the inequality chain

$$0 < \underline{2S < S + 1 < T} < 2 < 2T \tag{5.1}$$

is satisfied (which is the case in Region A of the (S, T)-phase plane).

5.1 Comparison to the evolutionary spatial prisoner's dilemma

An important preliminary result is established in this section which is utilised to facilitate an investigation into the ESS game dynamics in Region A of the (S, T)-phase plane. It is shown that the game dynamics of the ESS in Region A of the (S, T)-phase plane, shown in Figure 4.5, are identical to the game dynamics of the ESPD in a region where the latter game's parameter values satisfy $T + P \leq 2$.

Recall that the parameters in the normalised ESS and those in the normalised ESPD are $\Pi^S = \{T, 1, S, 0\}$ and $\Pi^{PD} = \{T, 1, 0, P\}$, respectively. The difference between these two parameter

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sets is that in the ESS, the punishment P for mutual defection takes the value 0 and the sucker's pay-off S assumes a value between 0 and 1, whereas in the ESPD, the punishment for mutual defection takes a value between 0 and 1, and the sucker's pay-off is prescribed a value of 0.

The achievable pay-off values for cooperators and defectors within each game are listed in Table 5.1 which, in combination with the inequalities defining the regions, yield the respective inequality chains

$$0 < 2S < S+1 < T \leq 2 < 2T \tag{5.2}$$

for the ESS, and

$$0 < 2P < 1 < T+P \leq 2 < 2T \tag{5.3}$$

or

$$0 < 1 < 2P < T+P \leq 2 < 2T \tag{5.4}$$

for the ESPD.

TABLE 5.1: *Achievable pay-off values in the ESPD and the ESS.*

	Cooperator			Defector		
Flanked by	D, D	C, D	C, C	D, D	C, D	C, C
ESS	$2S$	$S+1$	2	0	T	$2T$
ESPD	0	1	2	$2P$	$T+P$	$2T$

The two possible inequality chains of the ESPD arise because the region is specified only by the inequality $T \leq 2 - P$ and thus either (5.3) or (5.4) is possible. It is shown, however, that this has no bearing on the game dynamics.

Theorem 5.1 (Game dynamics of the ESS in Region A).
In the ESS on a cycle of order n and with pay-off parameters S and T, satisfying $T > S+1 > 2S$, the dynamics of the game mimic those of the ESPD on a cycle of order n and with pay-off parameter values T and P satisfying $T + P \leq 2$.

Proof: In the inequality chains (5.2), (5.3) and (5.4), the largest three values are awarded to the players with identical neighbourhood strategy compositions within their respective games, as may be verified in Table 5.1.

It remains to be verified that the game dynamics associated with the three smaller values of the inequality chain are identical regardless of the differences between their locations within the inequality chains and the make-up of the strategy neighbourhoods.

In either game, a cooperator playing against two defectors will defect during the next round. In the ESPD, this cooperator achieves a pay-off value of 0 compared with the defectors' pay-off values of at least $T + P$ each, while in the ESS, this cooperator obtains a pay-off value of $2S$ compared with the defectors' pay-off value of at least T each.

In either game, a defector playing against two defectors will defect during the following round as the players are not able to change strategies without witnessing an alternative strategy.

The behaviour of a cooperator playing against a cooperator and a defector depends on both the cooperator's neighbourhood and the defector's other neighbour. This cooperator is represented in boldface in the substate $X_\ell C\boldsymbol{C}DX_r$ and obtains a pay-off value of 1 in the ESPD and of $S + 1$ in the ESS. Therefore, the cooperator's pay-off value is smaller than that of the adjacent defector, which is at least $T + P$ in the ESPD and at least T in the ESS. The only case in which this cooperator will cooperate again during the following round is if the pay-off value of the adjacent cooperator is larger than the pay-off value of the adjacent defector, which is only

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the case for a pay-off value of 2 achieved by the adjacent cooperator (in both the ESPD and the ESS), and a pay-off value of $T + P$ in the ESPD or a pay-off value of T in the ESS for the adjacent defector. Thus, in order for the player represented in boldface to retain its strategy of cooperation, the only possible strategies for players X_ℓ and X_r are C and D, respectively, while any other configuration of strategies will result in the player represented in boldface adopting the strategy of defection during the following round. □

Having established that the game dynamics of the ESS are equivalent to those of the ESPD in the parameter regions where $2S < S + 1 < T$ and $T \leq 2 - P$, respectively, it is clear that the requirements for persistent cooperation, the probability of persistent cooperation in a randomly initialised game state and the enumeration of the components in the state graph in the case of the ESS are identical to their counterparts for the case of the ESPD described in [4]. The specific results are, however, restated in the remainder of this chapter for the sake of completeness.

5.2 Requirements for persistent cooperation

In the ESPD, and therefore, in Region A of the ESS, persistent cooperation takes the form of standoffs between runs of cooperators and defectors. These clusters must be of certain minimum requisite lengths in order to ensure a high enough pay-off value for the cooperator at the boundary and a low enough pay-off value for the defector at the boundary between runs of opposite strategy. The cooperator run is required to be of length at least 3 so that the cooperator at the boundary is witness to a pay-off value of 2, awarded to the central cooperator. The run of defectors must be of length at least 2 in order to ensure that the cooperator at the boundary compares that pay-off value of 2 with the defector's pay-off value of $T < 2$. These requirements are summarised in the following result.

Theorem 5.2 (Requirements for persistent cooperation, adapted from [4]). *In the ESS on a cycle with pay-off parameters S and T, satisfying $T > S + 1 > 2S$, only cooperation runs of length at least 3 can persist to the following round of the game and this happens if and only if they are flanked by two defection runs, each of length at least 2.*

The following theorem is concerned with the initial conditions required for persistent cooperation. These conditions include the substates described in Theorem 5.2 as well as substates that lead to these substates in certain worst-case scenarios. For example, the substate $\langle C\rangle^5$ leads to the substate $DD\langle C\rangle^3DD$ at some point if the defection runs adjacent to it grow in length.

Theorem 5.3 (Initial substates leading to persistent cooperation, adapted from [4]). *In the ESS on a cycle of order n with pay-off parameters S and T, satisfying $2S < S+1 < T$, an initial state will lead to persistent cooperation if and only if it contains any one or a combination of the states $\langle C\rangle^5$, $DD\langle C\rangle^3DD$, and $DD\langle C\rangle^4D$.*

5.3 Probability of persistent cooperation

The probability of persistent cooperation in the ESS, given a random assignment of initial player strategies along a cycle of order n, is simply the probability that this initial state contains at least one of the partial states described in Theorem 5.3. The following two results pertain the probability of persistent cooperation in the ESS on a cycle. The first describes the probability of persistent cooperation as a function of n and the second describes the limit of this probability as n tends toward infinity.

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Theorem 5.4 (Probability of persistent cooperation, adapted from [4]). *In the ESS on a cycle of order n with pay-off parameters S and T, satisfying $2S < S+1 < T$, the probability that a random distribution of strategies will lead to persistent cooperation is given by*

$$P_a(n) = 1 - a_n/2^n, \tag{5.5}$$

where the value of a_n is defined by the recurrence relation

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} - a_{n-6} + a_{n-7} \tag{5.6}$$

with seed values $a_1^* = 1$, $a_2^* = 3$, $a_3^* = 7$, $a_4^* = 15$, $a_5^* = 26$, $a_6^* = 45$, and $a_7^* = 99$.

The values of a_n and 2^n are tabulated in Table 5.2 for $n \in \{1, \dots, 15\}$ and the probability of persistent cooperation $P_a(n)$ is illustrated graphically in Figure 5.1 for $n \in \{1, \dots, 30\}$.

TABLE 5.2: *Values of a_n in (5.6) and 2^n for $n \in \{1, \dots, 15\}$ used to compute the probability $P_a(n) = 1 - a_n/2^n$ of persistent cooperation resulting from a randomly generated initial game state for the ESS on a cycle of order n, with pay-off parameter values satisfying $2S < S+1 < T$.*

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a_n	1	3	7	15	31	57	99	183	349	668	1 288	2 469	4 720	9 061	17 372
2^n	2	4	8	16	32	64	128	256	512	1 024	2 048	4 096	8 192	16 384	32 768

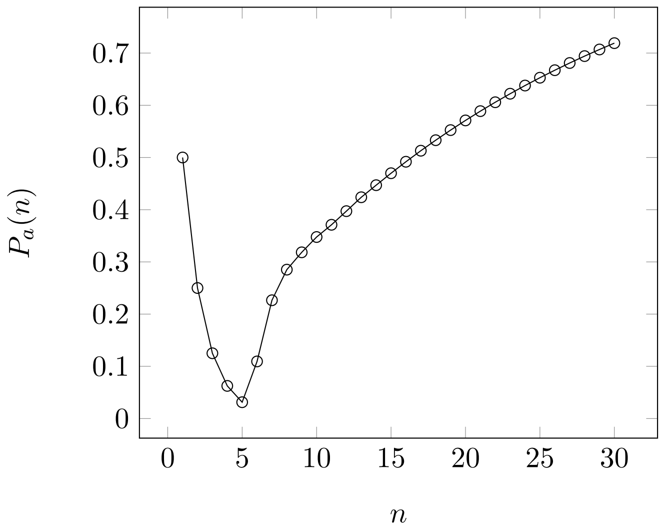

FIGURE 5.1: *The probability $P_a(n) = 1 - a_n/2^n$ of persistent cooperation in the ESS on a cycle of order n in Region A of the (S,T)-phase plane as a function of n.*

Closer inspection of the sequence a_n and the recurrence relation which defines it, as well as a comparison thereof with the sequence 2^n as n grows yields the final result of this section on the limit of the probability of persistent cooperation as the order of the underlying cycle tends to infinity.

Theorem 5.5 (Limit of probability of persistent cooperation, adapted from [4]). *In the ESS on a cycle of order n with pay-off parameters S and T, satisfying $2S < S+1 < T$, the probability $P_a(n)$ of persistent cooperation satisfies $\lim_{n\to\infty} P_a(n) = 1$.*

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5.4 State graph enumeration

This section pertains to the enumeration of the components in the state graph of the ESS in Region A of the (S, T)-phase plane. As in the investigation thus far, the result presented here follows immediately from Theorem 5.1 which reveals that an enumeration of the components in the state graph in the ESS on a cycle in Region A is the same as that for the ESPD on a cycle with parameters satisfying $T + P \leq 2$. As such, the result is restated from the investigation in [4]. The components of the state graph of the game are the components in which the steady states are the two homogeneous states, the all-cooperator and all-defector states, as well as the components in which the steady states comprise cooperation runs of length at least 3 and defection runs of length at least 2.

Theorem 5.6 (Enumeration of components in the state graph, adapted from [4]). *In the ESS on a cycle of order n with pay-off parameters S and T, satisfying $2S < S + 1 < T$, the state graph has*

$$Q^a(n) = 2 + \sum_{i=1}^{\lfloor \frac{n}{5} \rfloor} \frac{1}{2i} \left[\binom{n-3i-1}{2i-1} + \sum_{j \in \mathcal{W}} \binom{(n-3i)\mathrm{gcd}(i,j)/i - 1}{2\mathrm{gcd}(i,j) - 1} + i \sum_{k=0}^{\lfloor \frac{n-5i}{2} \rfloor} (n - 5i - 2k + 1) \binom{k+i-2}{i-2} \right] \tag{5.7}$$

components, where $\mathcal{W}$ is the set $\{x \in \mathbb{N} \mid i \text{ divides } n \cdot \mathrm{gcd}(i,j) \text{ and } x < i\}$. Each of these components contains only a single steady state. Steady states other than the all-cooperator steady state or the all-defector steady state are those states in which every cooperation run has length at least 3 and each defection run has length at least 2.

The number $Q^a(n)$ of components in the state graph of the ESS on a cycle in Region A of the (S, T)-phase plane is tabulated in Table 5.3 for $n \in \{1, \ldots, 37\}$ and is illustrated graphically in Figures 5.2 and 5.3 on linear and log-linear axes formats, respectively.

TABLE 5.3: *$Q^a(n)$, the number of components in the state graph of the ESS on a cycle of order $n \in \{1, \ldots, 37\}$ in Region A of the (S, T)-phase plane.*

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$Q^a(n)$	2	2	2	2	3	4	5	6	7	9	11	15	19	26

15	16	17	18	19	20	21	22	23	24	25	26	27
34	46	61	85	115	162	226	320	451	649	924	1 337	1 931

28	29	30	31	32	33	34	35	36	37
2 813	4 092	5 999	8 780	12 923	19 022	28 096	41 520	61 524	91 191

5.5 Fixation probabilities

Having defined the notion of a fixation probability in §4.5 as the probability that a mutant strategy, once introduced, establishes itself and grows in run size since its inception, this section is devoted to a brief investigation of the fixation probabilities in Region A of the (S, T)-phase plane. A group of mutant cooperators can never fix a population of defectors because of the growth requirement. The following lemma describes this formally.

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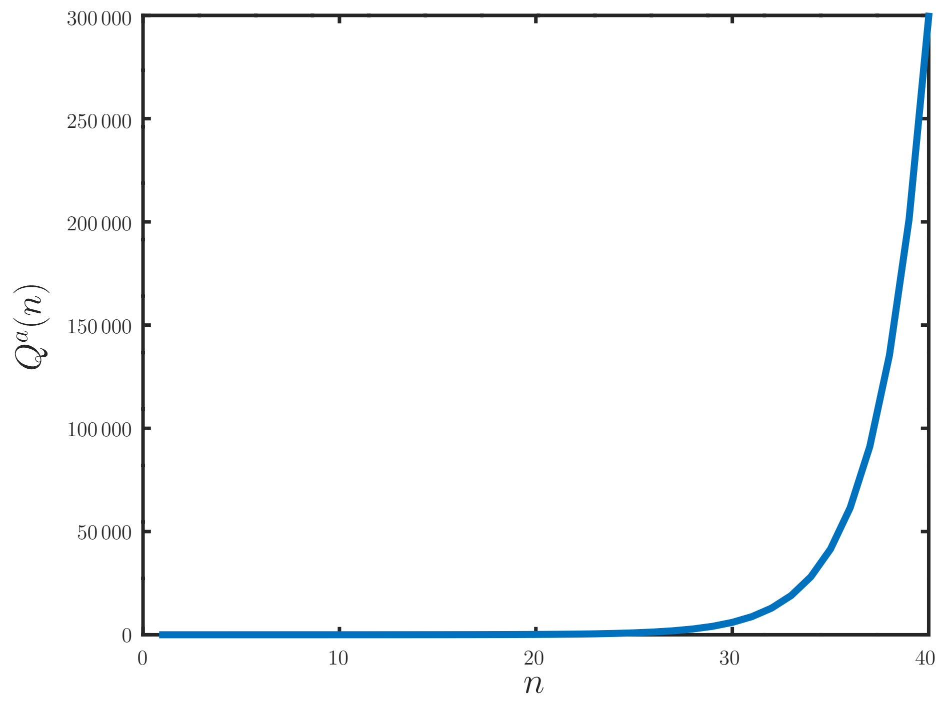

FIGURE 5.2: *The number of components $Q^a(n)$ in the state graph of the ESS on a cycle of order n in Region A of the (S, T)-phase plane, as enumerated in Theorem 5.6, on a linear vertical axis.*

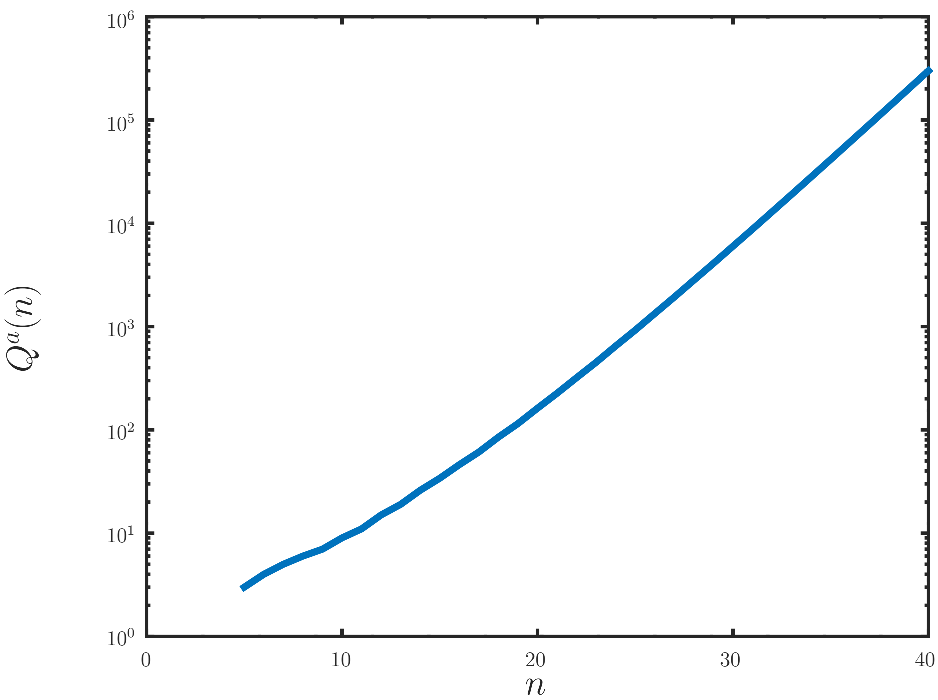

FIGURE 5.3: *The number of components $Q^a(n)$ in the state graph of the ESS on a cycle of order n in Region A of the (S, T)-phase plane, as enumerated in Theorem 5.6, on log-linear axes.*

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Lemma 5.1 (Fixation probability of the cooperation strategy).
In the ESS on a cycle of order n with pay-off parameters S and T, satisfying $2S < S+1 < T$, the fixation probability $F_a^C(n,k)$ of a group of k mutant cooperators among a population of $n-k$ defectors is zero.

Proof: The requirements for fixation of a population by a mutant strategy include growth of the mutant strategy amongst the players of the population. There is no substate in Region A of the (S,T)-phase plane in which a defecting player adopts the strategy of cooperation during any subsequent game round. This is clear from the inequality chain $2S < S+1 < T$ defining the region in which the smallest obtainable pay-off value of a defector adjacent to at least one cooperator is larger than the largest obtainable pay-off value of a cooperator adjacent to at least one defector. □

Next a lower bound on the fixation probability of a group of mutant defectors is established.

Lemma 5.2 (Fixation probability of the defection strategy).
In the ESS on a cycle of order n with pay-off parameters S and T, satisfying $2S < S+1 < T$, the fixation probability $F_a^D(n,k)$ of a group of k mutant defectors among a population of $n-k$ cooperators satisfies

$$F_a^D(n,k) > \frac{\binom{n-k}{k}}{\binom{n-1}{k}}. \tag{5.8}$$

Proof: In the proof of Lemma 5.1, it was shown that once a player has adopted the strategy of defection it will not return to the strategy of cooperation during any subsequent game round. This shows that a group of mutant defectors will certainly establish itself. Consider the likelihood that each mutating defector is located such that there are no defection runs of length at least 2 during the round in which the mutation occurs. This means that each mutant defector is a singleton and therefore all cooperators adjacent to a mutant adopts the strategy of defection during the following game round because of the defectors' pay-off value of $2T$ each, the highest obtainable in the game. After the initial mutation at least k cooperators will adopt the strategy of defection during the following round in such a scenario because each initially mutated defector is adjacent to two cooperators (which are possibly overlapping). The number of ways in which the k mutant defectors may be positioned as described above is $\binom{n-k}{k}$, which is the number of ways in which k objects may be distributed among $n-k$ containers without replacement (no container may have more than one object). The total number of arrangements of the k mutants among the $n-k$ cooperating players may be determined by a similar assignment of objects into containers, but this time with replacement, which is $\binom{n-k+k-1}{k}$, or equivalently $\binom{n-1}{k}$. The probability of the scenario described above is therefore

$$\frac{\binom{n-k}{k}}{\binom{n-1}{k}}.$$

This is, however, only a lower bound on the fixation probability of defection because in truth there only needs to be one such singleton defector for growth of the mutant strategy to occur. □

The results of Lemmas 5.1 and 5.2 is that the defection strategy is favoured above the strategy of cooperation in the ESS on cycle in Region A. Observing the game dynamics, this statement would seem intuitive as there is never any growth of the cooperation strategy while the strategy of defection is able to exhibit growth in some instances. It remains true that the probability of persistent cooperation in the ESS on a cycle in Region A of the (S,T)-phase plane is good and therefore cooperation is rarely eradicated entirely, indicating that population structure is an enabler of cooperation.

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5.6 Chapter summary

In this chapter, the region of the (S, T)-phase plane in which the pay-off parameter values satisfy $2S < S + 1 < T$ was analysed. The main finding, elucidated in §5.1, was that the ESS game dynamics are identical to the dynamics of the ESPD on a cycle of order n with its pay-off parameter values T and P satisfying $T + P \leq 2$. The results of the investigation into the ESPD on a cycle in [4] therefore hold true for the ESS in Region A as well. The requirements for persistent cooperation and subsequently the probability thereof from a randomly generated initial state were restated in §5.2 and §5.3, respectively. Enumeration of the components in the state graph of the ESS in Region A was discussed in §5.4. Finally, a brief investigation into the fixation probabilities of cooperation and defection in Region A followed in §5.5.

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CHAPTER 6

The case where $2S < T < S + 1$ (Region B)

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This chapter is devoted to a long-time asymptotic analysis of the ESS on a cycle in the case of Region B of the (S, T)-phase plane in Figure 4.5. The chapter opens in §6.1 with a number of preliminary results, after which the initial game states leading to some form of persistent cooperation are characterised in §6.2. This characterisation is used to derive the probability in §6.3 that a randomly generated initial state will lead to some form of persistent cooperation. Then a variation on the notion of a fixation probability is computed in §6.4. The chapter finally closes with an enumeration in §6.5 of the state graph components and with a brief summary of the chapter contents in §6.6.

All of the results established in this chapter therefore hold if the inequality chain

$$0 < \underline{2S < T < S + 1} < 2 < 2T \tag{6.1}$$

is satisfied (which is the case in Region B of the (S, T)-phase plane).

6.1 Preliminary results

A number of preliminary results are established in this section and are utilised in later sections of the chapter. The first result pertains to internal players of the ESS on a cycle. An *internal player* p is one that is flanked on both sides by players adopting the same strategy as that of p.

Theorem 6.1 (Internal players).
An internal player of the ESS on a cycle does not change its strategy from one game round to the next.

Proof: By contradiction. Suppose p is an internal player during round t of the ESS on a cycle and that p changes its strategy from round t to $t+1$ of the game. Then p is adjacent to a player

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q during round t which adopts the opposite strategy to that of p, and obtains a larger pay-off value than that of p. But this is a contradiction, because the strategies of p and q are identical during round t in view of the fact that p is an internal player during that round. $\square$

The next result establishes the fact that there can be no stand-offs along the boundaries between neighbouring runs of (different) strategies.

Theorem 6.2 (No stand-offs).
In the ESS on a cycle of order $n \geq 4$ with pay-off parameters S and T, satisfying $2S < T < S+1$, at least one player adjacent to the boundary between two adjacent runs of (different) strategies changes its strategy during the next game round.

Proof: By contradiction. Consider the partial game state $X_\ell CDX_r$. The four players adopting these strategies are referred to as the left-most player, the cooperator, the defector and the right-most player, from left to right. For a stand-off to occur between the two central players, the cooperator and the defector, one of the following cases must occur:

1. *The pay-off value of the cooperator equals that of the defector.* This is a contradiction, because no two pay-off parameters are equal in Region B of the (S, T)-phase plane.

2. *X_r=D and the pay-off value of the defector is smaller than that of the cooperator which is, in turn, smaller than that of the right-most player.* In this case, the pay-off value of the right-most player is at most T, because this player is adjacent to at least one other defector. Furthermore, the pay-off value of the cooperator is $S + 1$, since its pay-off value is larger than that of the defector. But then the pay-off value of the right-most player is $2T$, a contradiction.

3. *$X_\ell = C$ and the pay-off value of the cooperator is smaller than that of the defector which is, in turn, smaller than that of the left-most player.* In this case, the cooperator is adjacent to a defector and another cooperator, and so its pay-off value is $S + 1$. Furthermore, the play-off value of the defector is $2T$, because its pay-off value is greater than that of the cooperator. But then the pay-off value of the left-most player is more than $2T$, a contradiction, because the largest pay-off value obtainable by a cooperator is 2. $\square$

A necessary condition is next established for a defector changing its strategy to that of cooperation from one game round to the next.

Theorem 6.3 (A necessary condition for cooperation growth).
In the ESS on a cycle of order $n \geq 4$ with pay-off parameters S and T, satisfying $2S < T < S+1$, a defector during game round t only adopts the strategy of cooperation during game round $t + 1$ if it is adjacent to both a cooperator and a defector during round t, and if the latter cooperator is adjacent to another cooperator (during round t).

Proof: Consider the partial game state CD during game round t, let i denote the number of cooperators in the neighbourhood of the player adopting the strategy of cooperation, and let j denote the number of cooperators in the neighbourhood of the player adopting the strategy of defection. Then $i \in \{0, 1\}$ and $j \in \{1, 2\}$. Furthermore, the pay-off value of the player adopting the strategy of cooperation is $i + (2 - i)S$, while that of the player adopting the strategy of defection is jT. In order for the player adopting the strategy of defection to change its strategy during game round $t + 1$, it is therefore required that

$$i + (2 - 1)S > jT, \tag{6.2}$$

and that $j = 1$ (because a pay-off value of $2T$ is greater than all other pay-off values in Region B of the (S, T)-phase plane). If, however, $i = 0$, then it follows from (6.2) that $2S > T$,

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contradicting (6.1). Therefore, a necessary condition for the player adopting the strategy of defection during game round t changing its strategy to that of cooperation during game round $t+1$ is that $i = j = 1$. □

The following corollary follows immediately from the results above.

Corollary 6.1 (A sufficient condition for cooperation decline).
In the ESS on a cycle with pay-off parameters S and T, satisfying $2S < T < S+1$, any cooperator during game round t who is neither an internal player nor adjacent to another cooperator and to a defection run of length at least 2 adopts the strategy of defection during game round $t+1$.

6.2 Initial states leading to persistent cooperation

In the ensuing analysis it is important to bear in mind the relationships between the pay-off parameters and the relative sizes of the pay-off values achievable as a result of the inequality chain (6.1): The largest possible pay-off value is $2T > 2$ because of the inequalities $1 < T < 2$. The second largest pay-off value obtainable, 2, is that of a cooperator playing against two other cooperators. Lastly, the pay-off values $S+1$, T, $2S$ and 0 are strictly decreasing.

Lemma 6.1 (A sufficient condition for persistence).
In the ESS on a cycle of order $n \geq 4$ and with pay-off parameters S and T, satisfying $2S < T < S+1$, a run at least four cooperators will return to their strategy of cooperation at least every second subsequent round.

Proof: Consider a cooperation run $\boldsymbol{C} = \langle C \rangle^k$ of length $k \geq 4$, flanked by two defection runs $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$, during game round t. Two cases are considered:

Case 1: Both $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ have length 1. In this case, each of the outer cooperators defects during the following round by Corollary 6.1. During the round $t+1$, there are hence two cooperators, both of which are adjacent to defection runs of length at least 2, and so, by Theorem 6.3, all of the original players of $\boldsymbol{C}$ again cooperate during round $t+2$.

Case 2: At least one of $\boldsymbol{D}_1$ and/or $\boldsymbol{D}_2$ has length at least 2. Suppose, without loss of generality, that $\boldsymbol{D}_1$ has length at least 2. Then, by Theorem 6.3, the defector of $\boldsymbol{D}_1$ adjacent to $\boldsymbol{C}$ cooperates during round $t+1$ and there are at least k cooperators among the players of the adjacent runs $\boldsymbol{D}_1$ and $\boldsymbol{C}$. Moreover, during round $t+1$, the defection run $\boldsymbol{D}_1$ has either diminished in length to 1 or some length still at least 2. In both of these cases, the cooperator adjacent to $\boldsymbol{D}_1$ cooperates during round $t+2$ as it is flanked by two cooperators. Two subcases are now considered:

> Subcase 2.1: $\boldsymbol{D}_2$ originally also had length at least 2. In this case, the cooperator adjacent to $\boldsymbol{D}_2$ and the defector adjacent to $\boldsymbol{C}$ cooperate during round $t+1$. The cooperator in $\boldsymbol{C}$ adjacent to $\boldsymbol{D}_2$ therefore cooperates during round $t+2$ as it was flanked by two cooperators during round $t+1$.
>
> Subcase 2.2: $\boldsymbol{D}_2$ originally had length 1. Thus, the cooperator adjacent to $\boldsymbol{D}_2$ defects during round $t+1$ along with the defector in $\boldsymbol{D}_2$. During round $t+1$, these two players form part of a defection run of length 2 adjacent to a cooperation run of length at least 2 and so, by Theorem 6.3, the original outer cooperator of $\boldsymbol{C}$ cooperates during round $t+2$.

In each of the aforementioned cases, each player in $\boldsymbol{C}$ returns to the strategy of cooperation during round $t+2$ and although the defection runs flanking $\boldsymbol{C}$ may possibly have changed in length, they still conform to either Case 1 or Case 2 above. □

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From the above lemma, it is clear that any initial state containing a run of at least four cooperators will result in persistent cooperation as the players in this run will all cooperate during at least every second subsequent round and thus not all defect during any game round. Another initial state which necessarily results in persistent cooperation is identified in the following lemma.

Lemma 6.2 (Another sufficient condition for persistence).
In the ESS on a cycle of order $n \geq 4$ and with pay-off parameters S and T, satisfying $2S < T < S + 1$, any four adjacent players with strategies $CCDD$ will indefinitely have at least two adjacent cooperators among them.

Proof: Consider first the player interaction in the interior of the substate $\boldsymbol{R} = CCDD$. By Theorem 6.3, the inner defector cooperates during round $t + 1$. Three cases are considered:

Case 1: The cooperators in $\boldsymbol{R}$ are adjacent to another cooperator outside $\boldsymbol{R}$. In this case, the outer cooperator of $\boldsymbol{R}$ cooperates during round $t + 1$.

Case 2: The cooperators in $\boldsymbol{R}$ are adjacent to a run of at least two defectors outside $\boldsymbol{R}$. In this case, the defector adjacent to $\boldsymbol{R}$ cooperates during round $t + 1$. Therefore, the cooperation run grows in length from round t to round $t + 1$ and hence $\boldsymbol{R}$ retains at least two adjacent cooperators.

Case 3: The cooperators in $\boldsymbol{R}$ are adjacent to a singleton defector outside $\boldsymbol{R}$. In this case, the outer cooperator of $\boldsymbol{R}$ and the singular defector to which it was adjacent defect during round $t + 1$ by Corollary 6.1. During round $t + 1$, there are two cooperators (one from the original pair within $\boldsymbol{R}$ as well as the inner defector from $\boldsymbol{R}$) adjacent to a defection run of length at least 2. By Theorem 6.3, the original cooperators of $\boldsymbol{R}$ again cooperate during round $t + 2$. □

The lemmas presented thus far in this section establish sufficient initial state conditions for persistent cooperation. The lemma that follows will be used to show that these conditions are indeed also necessary for persistent cooperation and hence that no other initial substates can result persistent cooperation.

Lemma 6.3 (The behaviour of singleton players).
In the ESS on a cycle of order $n \geq 3$ and with pay-off parameters S and T, satisfying $2S < T < S + 1$, any combination (possibly overlapping) of only the substrings CDC and DCD will necessarily result in the entire ensemble defecting during the next game round.

Proof: In each instance of the substate CDC, each cooperator obtains a pay-off value of at most $S + 1$ while the defector obtains a pay-off value of $2T$. The cooperators therefore necessarily defect during the next game round while the defector retains its strategy.

Similarly, in any instance of the substate DCD, the cooperator obtains a pay-off value of $2S$ while the defectors each obtains a pay-off value of at least T. The cooperator therefore defects during the following game round. Two cases are finally considered to show that the defectors in the substate DCD do not change their strategy to cooperation during the following game round:

Case 1: Such a defector is adjacent to a cooperator outside of the substate DCD. In this case, the pay-off value obtained by the defector in question is the largest pay-off value achievable, $2T$, and so the defector retains its strategy.

Case 2: Such a defector is adjacent to a defector outside the substate DCD. In this case, the pay-off value of the defector in question remains T, while the only cooperator adjacent to it achieves a pay-off value of $2S < T$. Therefore, the defector retains its strategy during the following round. □

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The proof of the following characterisation of initial game states which lead to persistent cooperation requires the notion of a pair of players: A *pair* of players during some game round is simply two adjacent players adopting the same strategy during that game round.

Theorem 6.4 (Characterisation of initial states leading to persistent cooperation). *In the ESS on a cycle of order $n \geq 4$ and with pay-off parameters S and T, satisfying $2S < T < S+1$, cooperation persists if and only if the initial state contains either (or both) of the substates $CCCC$ or $CCDD$.*

Proof: It follows from Lemmas 6.1 and 6.2 that should either (or both) of the substates $CCCC$ or $CCDD$ be present in the initial state, cooperation persists. It therefore remains to be shown that no other state leads to persistent cooperation.

It follows from Lemma 6.3 that any initial state consisting only of combinations of the substrings CDC and DCD (possibly with overlapping) leads to the all-defector steady state. Initial states not consisting solely of combinations of the substrings CDC and DCD (possibly with overlapping) necessarily contain one or more of the following substates:

1. $CCDD$ or $DDCC$,
2. a run of at least four cooperators,
3. a run of three cooperators flanked by singleton defectors, or
4. a run of at least three defectors flanked by singleton cooperators.

Cases 1 and 2 above have already been shown to lead to persistent cooperation

In case 3 above, the initial state contains the substate $CDCCCDC$. In this substate, there are two instances of the substate CDC with an additional cooperator between them. By Lemma 6.3, each member of the substates CDC defects, while the internal cooperator retains its strategy of cooperation during round 2. This leads to the substate $DDDCDDD$ during round 2. By Corollary 6.1, the cooperator defects during round 3.

In case 4 above, the initial state contains the substate $DC\langle D\rangle^k CD$ for some $k \geq 3$. This is equivalent to two instances of the substate DCD with the substate $\langle D\rangle^{k-2}$ between them. By Lemma 6.3, each player in the instances of the substate DCD defects during round 2, while the players in the susbtate $\langle D\rangle^{k-2}$ retain the strategy of defection during the following round. $\square$

The requirement for persistent cooperation in the ESS on a cycle of order $n \geq 4$ in Region B of the (S,T)-phase plane is therefore the presence of either of the states $CCCC$ and/or $CCDD$ in the initial state of the game and there are no other substates that can lead to persistent cooperation.

6.3 The probability of persistent cooperation

In this section, the requirements for persistent cooperation established in Theorem 6.4 are used in conjunction with the transfer matrix method described in §2.3 in order to enumerate the initial states that do not lead to persistent cooperation (*i.e.* initial state strings that contain neither of the substrings required for persistent cooperation). In order to do this, a digraph, denoted by D_3, is constructed in which each vertex v_i represents one of the possible binary strings of length 3. A vertex v_i representing the string $s_1s_2s_3$ is incident to a vertex v_j representing the string

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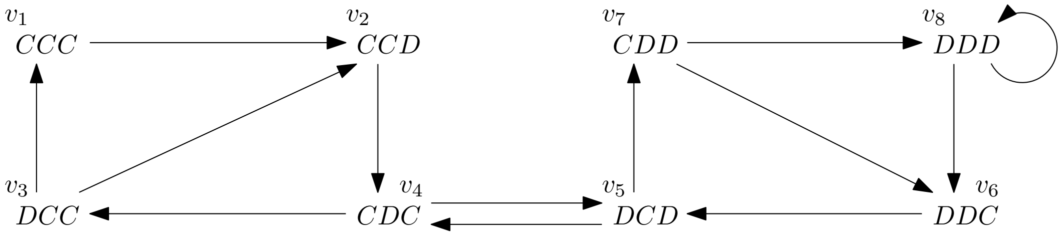


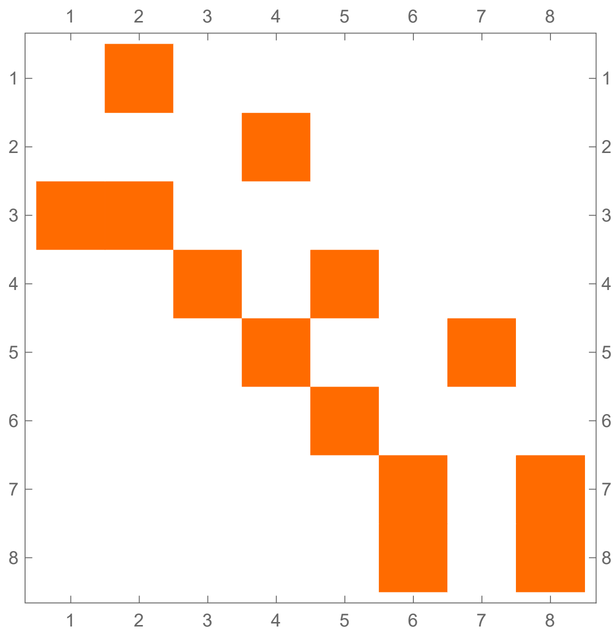
FIGURE 6.1: *A digraph, D_3, required during the enumeration of strings of length n that contain neither of the strings $CCDD$ nor $CCCC$ in the characterisation of Theorem 6.4.*

$s_2s_3s_4$ if and only if the string $s_1s_2s_3s_4$ contains neither of the substrings $CCDD$ and/or $CCCC$ required for persistent cooperation. The digraph D_3 is represented graphically in Figure 6.1.

The structure of the adjacency matrix of D_3, denoted by $\boldsymbol{B}$, is depicted in Table 6.1. Since the determinant $\det(\boldsymbol{I} - x\boldsymbol{B})$ is $1 - x - x^2 - x^3 + x^5 + x^6 + x^7$, the number b_n of initial game states that contain neither of the substrings required for persistent cooperation on a cycle of order n, as characterised in Theorem 6.4, satisfies the recurrence relation

$$b_n = b_{n-1} + b_{n-2} + b_{n-3} - b_{n-5} - b_{n-6} - b_{n-7}. \tag{6.3}$$

TABLE 6.1: *Structure visualisation of the adjacency matrix $\boldsymbol{B}$ of the digraph D_3 in which ones are indicated by filled blocks while zeros are left blank.*



This recursive relation (6.3) requires seed values $b_1^*, \ldots, b_7^*$ in order to facilitate calculation of the values b_8, b_9, b_{10}, ... These seed values are the coefficients of the Maclaurin expansion of

$$-x\frac{-1 - 2x - 3x^2 + 5x^4 + 6x^5 + 7x^6}{1 - x - x^2 - x^3 + x^5 + x^6 + x^7}, \tag{6.4}$$

which is given by

$$x + 3x^2 + 7x^3 + 11x^4 + 16x^5 + 27x^6 + 48x^7 + 75x^8 + \ldots \tag{6.5}$$

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The seed values for (6.3) are therefore $b_1^* = 1$, $b_2^* = 3$, $b_3^* = 7$, $b_4^* = 11$, $b_5^* = 16$, $b_6^* = 27$ and $b_7^* = 48$. Note that these values are not the numbers of initial states that lead to the all-defection steady states on cycles of orders 1–7, but are simply seed values for the recursive expression of b_n for $n \geq 8$.

Using these seed values in conjunction with the recursive expression (6.3), numerical values for b_8, b_9, b_{10}, ... may be determined. Considering that b_n denotes the number of possible initial states that lead to the all defector steady state, and that the total number of possible initial states is 2^n, the probability of all players eventually defecting from a randomly generated initial game state may be calculated as $\frac{b_n}{2^n}$. Taking the complement, the probability of persistent cooperation in the ESS on a cycle of order n in Region B of the (S, T)-phase plane of Figure 4.5, is given by

$$P_b(n) = 1 - \frac{b_n}{2^n}. \tag{6.6}$$

This probability is plotted against n, the order of the underlying cycle, in Figure 6.2. It can be seen in the figure that this probability is increasing.

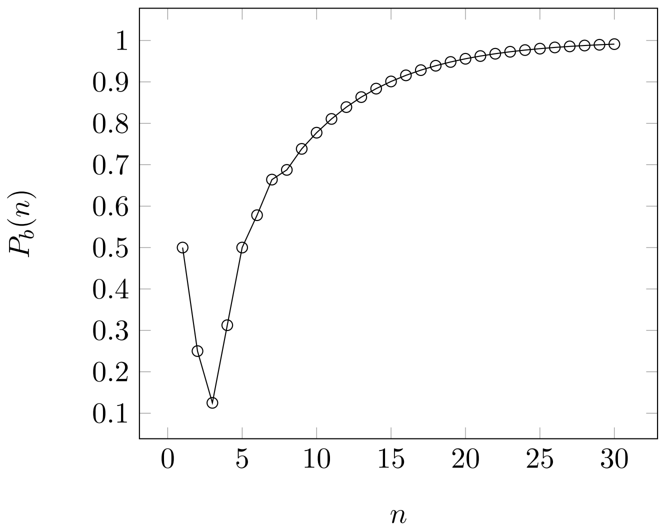

FIGURE 6.2: *The probability $P_b(n) = 1 - b_n/2^n$ of persistent cooperation in the ESS on a cycle of order n in Region B of the (S, T)-phase plane as a function of n.*

In Table 6.2, which contains the values of b_n and 2^n for $n \in \{1, \dots, 15\}$, it can further be seen that the values of b_n are increasing in n, which is important as this is not immediately obvious from the recurrence relation (6.3). Furthermore, the sequence b_n is increasing by some fraction that appears to be greater than 1. The following lemma, which clarifies that the sequence b_n is indeed increasing, establishes the limit of the sequence $b_1, b_2, b_3, \dots$ as n grows to infinity.

TABLE 6.2: *Values of b_n and 2^n used to compute the probability $P_b(n) = 1 - b_n/2^n$ of persistent cooperation resulting from a randomly generated initial game state on a cycle of order n, with pay-off parameter values satisfying $2S < T < S + 1$.*

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
b_n	1	3	7	11	16	27	48	80	134	228	388	659	1 120	1 905	3 242
2^n	2	4	8	16	32	64	128	256	512	1 024	2 048	4 096	8 192	16 384	32 768

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Lemma 6.4.
The sequence $b_1, b_2, b_3, \ldots$ satisfying the recurrence relation

$$b_i = b_{i-1} + b_{i-2} + b_{i-3} - b_{i-5} - b_{i-6} - b_{i-7}, \qquad i = 8, 9, 10, \ldots \tag{6.7}$$

with seed values $b_1 = 1$, $b_2 = 3$, $b_3 = 7$, $b_4 = 11$, $b_5 = 16$, $b_6 = 27$ and $b_7 = 48$, is strictly increasing.

Proof: It is shown by the strong form of induction that $b_i \geq \frac{6}{5} b_{i-1}$ for any natural number $i \geq 2$. Note, as base case for the induction process, that the statement is true for the seed values $b_2, \ldots, b_7$ and assume, as induction hypothesis, that it holds for all $i \in \{2, \ldots, k\}$, where k is some integer. Note also that $b_i \geq 0$ for all $i \in \{2, \ldots, k-1\}$ by the induction hypothesis since $b_1 = 1$. It follows from (6.7) and repeated use of the inequality $-b_{i-1} \geq -\frac{1}{a} b_i$ (for $i \leq k$ and $a = \frac{6}{5}$) that

$$\begin{aligned}
b_{k+1} &= b_k + b_{k-1} + b_{k-2} - b_{k-4} - b_{k-5} - b_{k-6} \\
&\geq b_k + b_{k-1} + b_{k-2} - b_{k-4} - b_{k-5} - \frac{b_{k-5}}{a} \\
&\geq b_k + b_{k-1} + b_{k-2} - b_{k-4} - \frac{a+1}{a^2} b_{k-4} \\
&\geq b_k + b_{k-1} + b_{k-2} - \frac{a^2+a+1}{a^3} b_{k-3} \\
&\geq b_k + b_{k-1} + b_{k-2} - \frac{a^2+a+1}{a^4} b_{k-2} \\
&\geq b_k + b_{k-1} - \frac{-a^4+a^2+a+1}{a^5} b_{k-1} \\
&\geq b_k - \frac{-a^5-a^4+a^2+a+1}{a^6} b_k \\
&= -\frac{-a^6-a^5-a^4+a^2+a+1}{a^6} b_k.
\end{aligned}$$

Therefore,

$$b_{k+1} \geq \frac{(\frac{6}{5})^6 + (\frac{6}{5})^5 + (\frac{6}{5})^4 - (\frac{6}{5})^2 - (\frac{6}{5}) - 1}{(\frac{6}{5})^6} b_k,$$

which simplifies to $b_{k+1} \geq 1.309 b_k \geq \frac{6}{5} b_k$. $\square$

The fact that the sequence $P_b(1)$, $P_b(2)$, $P_b(3), \ldots$ is increasing for all $n \geq 8$ may be leveraged to show that the limit of $P_b(n)$ as $n \to \infty$ is unity.

Theorem 6.5.
The probability $P_b(n)$ that a randomly generated initial state of the ESS on a cycle of order n with pay-off parameter values satisfying $2S < T < S+1$ results in some form of persistent cooperation satisfies

$$\lim_{n\to\infty} P_b(n) = \lim_{n\to\infty} \left(1 - \frac{b_n}{2^n}\right) = 1.$$

Proof: Setting $L_n = b_{n-2} + b_{n-3} - b_{n-5} - b_{n-6} - b_{n-7}$ yields $b_n = b_{n-1} + L_n$. Subtracting L_n from b_{n-1} therefore gives

$$\begin{aligned}
b_{n-1} - L_n &= b_{n-2} + b_{n-3} + b_{n-4} - b_{n-6} - b_{n-7} - b_{n-8} - (b_{n-2} + b_{n-3} - b_{n-5} - b_{n-6} - b_{n-7}), \\
&= b_{n-4} + b_{n-5} - b_{n-8}.
\end{aligned}$$

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It follows from Lemma 6.4 that $b_{n-4} + b_{n-5} - b_{n-8} > 0$ and hence that $b_{n-1} > L_n$. This means that $0 < b_n < 2b_{n-1}$. Dividing this inequality chain right through by 2^n yields

$$0 < \frac{b_n}{2^n} < \frac{b_{n-1}}{2^{n-1}},$$

from which it follows that the sequence $\frac{b_8}{2^8}, \frac{b_9}{2^9}, \frac{b_{10}}{2^{10}}, \ldots$ remains positive and is strictly decreasing. The Monotonic Sequence Theorem [32] guarantees convergence of the sequence under these conditions.

Having established that the sequence converges, denote its limiting value by

$$\lim_{n\to\infty} \frac{b_n}{2^n} = V. \tag{6.8}$$

Then it follows from (6.3) that

$$\begin{aligned}
V &= \lim_{n\to\infty} \frac{b_{n-1} + b_{n-2} + b_{n-3} - b_{n-5} - b_{n-6} - b_{n-7}}{2^n} \\
&= \lim_{n\to\infty} \frac{b_{n-1}}{2^n} + \lim_{n\to\infty} \frac{b_{n-2}}{2^n} + \lim_{n\to\infty} \frac{b_{n-3}}{2^n} - \lim_{n\to\infty} \frac{b_{n-5}}{2^n} - \lim_{n\to\infty} \frac{b_{n-6}}{2^n} - \lim_{n\to\infty} \frac{b_{n-7}}{2^n} \\
&= \frac{1}{2} \lim_{n\to\infty} \frac{b_{n-1}}{2^{n-1}} + \frac{1}{2^2} \lim_{n\to\infty} \frac{b_{n-2}}{2^{n-2}} + \frac{1}{2^3} \lim_{n\to\infty} \frac{b_{n-3}}{2^{n-3}} \\
&\quad - \frac{1}{2^5} \lim_{n\to\infty} \frac{b_{n-5}}{2^{n-5}} - \frac{1}{2^6} \lim_{n\to\infty} \frac{b_{n-6}}{2^{n-6}} - \frac{1}{2^7} \lim_{n\to\infty} \frac{b_{n-7}}{2^{n-7}} \\
&= \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^5} - \frac{1}{2^6} - \frac{1}{2^7}\right) \lim_{n\to\infty} \frac{b_n}{2^n}. \\
&= \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^5} - \frac{1}{2^6} - \frac{1}{2^7}\right) V,
\end{aligned}$$

which implies that $V = 0$ and consequently $\lim_{n\to\infty} P_b(n) = 1 - 0 = 1$. □

6.4 Fixation probabilities

In this section, variations on the notion of a fixation probability are determined for the ESS on a cycle of order n in Region B of the (S, T)-phase plane. It follows from Lemma 6.2 that a pair of cooperators is required in order for a run of cooperation to grow amongst runs of defectors. The fixation probability of a singular cooperator invading a population of $n-1$ defectors in the ESS on a cycle of order n is therefore zero.

Consider now the variation on the notion of a fixation probability, in which two cooperators are introduced among a population of $n-2$ defectors at two random locations on the cycle of order $n \geq 5$. These two cooperators will cause a growth in the strategy of cooperation if they form a run of length 2. If the order of the underlying cycle is odd, the run of (initially) two cooperators grows over successive rounds until one defector remains, which in that round obtains the highest possible pay-off value and therefore grows its run of defection by 2 during the subsequent round. Therefore, the two new defectors will change their strategy back to cooperation during the following round. This phenomenon will repeat indefinitely and is called an *oscillation cluster*. If the order of the cycle is even, however, the run of cooperation will grow by 2 during each round until it has taken over the entire cycle. The following lemma captures this behaviour formally.

Lemma 6.5 (Oscillation clusters).
In the ESS on a cycle of order n with pay-off parameters S and T, satisfying $2S < T < S + 1$,

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an adjacent pair of cooperators in a cycle of $n-2$ defectors causes their cooperation run to grow by 2 during each round until either all players adopt the strategy of cooperation, or a singular defector remains. In the latter case, the defection run alternates between having length 1 and length 3 indefinitely.

Proof: This situation is a special case of the second case in the proof of Lemma 6.2, in which the defectors are all part of the same run. The cooperators' pay-off values of $S+1$ thus trump the pay-off values obtained by the two defectors directly adjacent to the cooperators ($T < S+2$). These defectors therefore cooperate during the subsequent game rounds. Two cases are considered:

Case 1: The order of the cycle is even. In this case, the cooperation cluster continues to grow by two during each game round until the cooperation run spans the entire cycle.

Case 2: The order of the cycle is odd. In this case, the cooperation run continues to grow by two during each game round until a singleton defector remains in the defection run during some round t. This defector obtains the pay-off value $2T$ during round t, while the adjacent cooperators each obtains a pay-off value of $S + 1 < 2T$ and so the adjacent cooperators defect during round $t + 1$. During round $t + 1$, however, the defection run again is of length 3, and so the cooperation run grows by two during round $t + 2$ while the singular defector remains. This group of three players centered on the singular defector therefore form an oscillation cluster. □

The above result is used in the proof of the following theorem which describes a special case of the fixation probability of cooperation for Region B. Note that the fixation of cooperation does not require elimination of the defection strategy in this case, since the remaining oscillation clusters are not considered as interfering with the fixation of the cooperation strategy.

Theorem 6.6 (Fixation probability of cooperation for $k = 2$).
In the ESS on a cycle of order $n \geq 5$ with pay-off parameters S and T, satisfying $2S < T < S + 1$, the fixation probability of two mutant cooperators, denoted by $F_b^C(n, 2)$, in a population of defectors is given by

$$F_b^C(n, 2) = \frac{2}{n-1}. \tag{6.9}$$

Proof: If the two cooperators are adjacent to one another, then the probability of fixation is 1 because, by Lemma 6.5, a pair of cooperators adjacent to a single run of defection grows until either a singular defector remains or until the entire population is cooperating (the latter being the desired result for fixation).

The probability of the two cooperators being placed adjacent to one another (thus forming a run of length 2) is $\frac{2}{n-1}$, as once the first cooperator is placed in any position of the cycle, there are 2 positions out of the possible $n - 1$ which may be chosen for the location of the second cooperator.

If the cycle order n is even, the cooperation grows to span the entire cycle. If, however, the order of the cycle is odd, the resulting oscillation cluster will remain indefinitely but does not grow to a length greater than 3 defectors during any future round, and so the conditions for the fixation probability defined in §4.5 are met. □

The situation of cooperation runs occurring among large defection runs is considered in the following lemma.

Lemma 6.6 (Formation of oscillation clusters).
Consider the ESS on a cycle of order $n \geq 5$ with pay-off parameters S and T, satisfying $2S < T < S + 1$. If every cooperation run has length at least 2 and every defection run has length at least 2, each cooperation run grows by 2 during every game round as long as their adjacent defector runs each has length at least 2. The defection runs of even length, shrink by 2 during

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each round until they disappear, while defection runs of odd length shrink until they have length 1, at which point they form an oscillation cluster.

Proof: It follows from Lemma 6.2 that as long as each defection run has length at least 2, every adjacent cooperation run grows by 2 during each game round. Each defection run of odd length therefore shrinks until it has length 1 during some round t (not necessarily the same round for each defection run). Such a singleton defector obtains the largest pay-off value $2T$ during game round t and so the adjacent cooperators defect during the following round. The cooperation runs of which the cooperators were members each has length at least 4 during round t (each cooperation run grew by 2 during the first game round and thus cannot be smaller than 4 during round t) and therefore has length at least 2 during round $t+1$. The singleton defector during round t and its two adjacent players hence form an oscillation cluster. □

Game states that contain oscillation clusters cannot enter a steady state, due to the dynamic nature of an oscillation cluster having at least two players alternating their strategies during each round. Once all of the oscillation clusters have formed, it is clear that only two game states will remain and that the game instance will oscillate between them. In the setting of a state graph, the remaining states through which a game instance cycles is called a *limit cycle.*

The result of Lemma 6.6 is developed further in order to show that all initial states containing either the substate $CCDD$ or the substate $CCCC$ enter a limit cycle in which the cooperation strategy is predominant and defection remains only in the form of oscillation clusters.

Theorem 6.7 (Near eradication of the defection strategy).
In the ESS on a cycle of order $n \geq 5$ with pay-off parameters S and T, satisfying $2S < T < S+1$, each initial state containing at least one of the substates $CCDD$ or $CCCC$ mentioned in Theorem 6.4 eventually passes through a state fulfilling the requirements of Lemma 6.6 or directly enter a limit cycle and, therefore, all defection will either be eliminated or remain only in the form of oscillation clusters.

Proof: Every player initially forming part of instances (possibly overlapping) of the substates CDC, DCD or $DC\langle D\rangle^k CD$ will defect during the second game round by Lemma 6.3, forming defection runs of length at least 3 during round 2. Furthermore, every player forming part of an instance of the state $CDCCCDC$ will defect by the latest during round 3, forming a defection run of length at least 3. Therefore, every player neither forming part of a substate $CCDD$ nor a substate $CCCC$ will have formed part of a defection run of length at least 2 by round 3 at the latest, unless it was already contained in an oscillation cluster. Therefore the game will reach a state satisfying the conditions in Lemma 6.6. □

Lemmas 6.5 and 6.6 and Theorem 6.7 show that in the presence of persistent cooperation the trend is for the cooperation strategy to grow within the cycle and for the defection strategy to be eradicated (with exception of oscillation clusters). Therefore, the fixation probability of the cooperation strategy is refined to the probability of persistent cooperation given fixed content in the context of a population adopting defection and a group of mutant cooperating players. The fixation probability of defection, on the other hand, is refined to the probability of nonpersistent cooperation (eradication of the cooperation strategy) given fixed content in the context of a population adopting the strategy of cooperation and a group of mutant defecting players.

The next result provides an upper bound on the fixation probability $F_b^C(n,k)$.

Theorem 6.8 (An upper bound on the fixation probability of cooperation).
In the ESS on a cycle of order $n \geq 5$ with pay-off parameters S and T, satisfying $2S < T < S+1$,

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the fixation probability of a group of mutant cooperators of size k satisfies

$$F_b^C(n,k) \leq 1 - \frac{\binom{n-k}{k}}{\binom{n-1}{k}}. \tag{6.10}$$

Proof: For cooperation to persist, it follows from Lemma 6.2 that a cooperation run of length at least 2 is required. Thus, for cooperation to grow, at least one cooperation run of length at least 2 is required. The probability of fixation is therefore bounded from above by the probability that two mutant players are adjacent to one another as this will result in either the all-cooperator steady state or a limit cycle containing at most $n/4$ oscillation clusters and therefore at most $n/4$ permanent defectors.

This probability is the complement of the probability that no pair of mutants is adjacent to one another. The number of ways in which the k mutants may be placed among $n-k$ defectors with no pairs of mutants adjacent to one another is the same as the number of ways k objects can be distributed among $n-k$ containers without replacement, namely $\binom{n-k}{k}$. The total number of ways in which the k mutants may be distributed among the $n-k$ defectors may be modelled by a similar assignment of objects into containers, but this time with replacement, which is $\binom{n-k+k-1}{k}$,or equivalently, $\binom{n-1}{k}$. The probability that no two mutants are adjacent to one another is therefore

$$\frac{\binom{n-k}{k}}{\binom{n-1}{k}}$$

and the probability that at least one pair of mutants is adjacent to one another is one less the value above. Furthermore, as soon as $k > n-k$, the probability of at least one pair of cooperators being adjacent is 1, by the pigeonhole principle.

This condition is, however, not sufficient for fixation to occur as there are special mutant placement patterns which counteract the growth of the original pair of cooperator mutants. For example, for $k \geq 4$, the pattern between two defection runs $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$, both of length at least 2, $\boldsymbol{D_1}CDCCDC\boldsymbol{D_2}$, clearly has a pair of cooperators adjacent to one another, yet will not result in growth of the defection strategy. The possibility of this and other patterns means that the probability of two adjacent mutants only serves as an upper bound on the fixation probability of the cooperation strategy. $\square$

The next result provides a lower bound on the fixation probability $F_b^C(n,k)$.

Theorem 6.9 (A lower bound on the fixation probability of cooperation). *In the ESS on a cycle of order $n \geq 5$ with pay-off parameters S and T, satisfying $2S < T < S+1$, the fixation probability of a group of mutant cooperators of size k satisfies*

$$F_b^C(n,k) \geq 1 - \frac{\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}\binom{n-4i-1}{k-4i}}{\binom{n-1}{k}}. \tag{6.11}$$

Proof: An occurrence of the substate $CCCC$ guarantees persistent cooperation by Lemma 6.1. It also guarantees that the game results in either the all-cooperator steady state or a limit cycle with oscillation clusters. An occurrence of the substate $CCCC$ therefore guarantees fixation of the cooperation strategy.

In order to determine the probability of at least one occurrence of the substate $CCCC$, when placing k cooperators randomly among a cycle of $n-k$ defectors, consider the probability of no run of at least four cooperators resulting. This probability may be found by means of the generating function for the number of non-negative integer solutions to the equation

$$x_1 + x_2 + x_3 + \cdots + x_{n-k} = k, \tag{6.12}$$

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with $x_1, \dots, x_{n-k} \leq 3$. The generating function for this problem is given by

$$\left[z^0 + z^1 + z^2 + z^3\right]^{n-k} = (1 - z^4)^{n-k}\left(\frac{1}{1-z}\right)^{n-k}. \tag{6.13}$$

For all n and k, the coefficient of the term z^k in the expansion of (6.13) is the number of non-negative integer solutions to (6.12). This coefficient is

$$\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}\binom{n-4i-1}{k-4i}. \tag{6.14}$$

The probability of not having any run of at least four cooperators is given by this term divided by the number of distributions of k cooperators among $n-k$ positions with replacement (the total number of possible placements without restriction),

$$\frac{\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}\binom{n-4i-1}{k-4i}}{\binom{n-1}{k}}. \tag{6.15}$$

The complement of the quantity in (6.15) is the probability that there is at least one run of at least four cooperators. □

The lower bound of Theorem 6.9 and the upper bound of Theorem 6.8 on the fixation probability of k mutant cooperators are plotted in Figure 6.3 for $n \in \{20, \dots, 50\}$ and $k \in \{4, \dots, 20\}$. Note that the lower bound was calculated using the probability of an occurrence of the substate $CCCC$, the rarer of the two conditions for persistent cooperation, and that the cases satisfying the requirements for the upper bound which violate the requirements for persistent cooperation are few and far between as for each pair of cooperators there have to be exactly two cooperators placed precisely in two out of the possible $n-k$ available locations along the cycle in order to realise the substate $DCDCCDCD$, avoiding both the substates $CCDD$ and $DDCC$.

Having calculated upper and lower bounds on the fixation probability of the strategy of cooperation for the ESS on a cycle, the focus now turns to the fixation probability of the strategy of defection for the ESS on a cycle.

Lemma 6.7 (The fixation probability of defection for small k).
In the ESS on a cycle of order $n \geq 5$ with pay-off parameters S and T, satisfying $2S < T < S+1$, the fixation probability of a group of mutant defectors of size $k < n/4$ is zero.

Proof: For fixation of the strategy of defection, each defector adjacent to a pair of cooperators has to be a singleton and each cooperator adjacent to a pair of defectors has to be a singleton by Corollary 6.1. The minimum number of defectors required in order to meet these conditions is given by $n/4$ as exemplified by states of the form

$$CCCDCCCD \cdots CCCD,$$

in which there is one defector for each triple of cooperators and for which fixation of the strategy of defection will occur by the third round. Until this occurs, however, it follows from the pigeonhole principle there will be at least one cooperation run of length at least 4, which indicates that the strategy of cooperation will remain ubiquitous with the possible exception of the formation of oscillation clusters, by Theorem 6.7. □

The next results provides an upper bound on the fixation probability of the strategy of defection in cases where $k \geq n/4$ for the ESS on a cycle of order n and a group of defecting mutants of size k. This serves to show that the upper bound on the fixation probability of defection is

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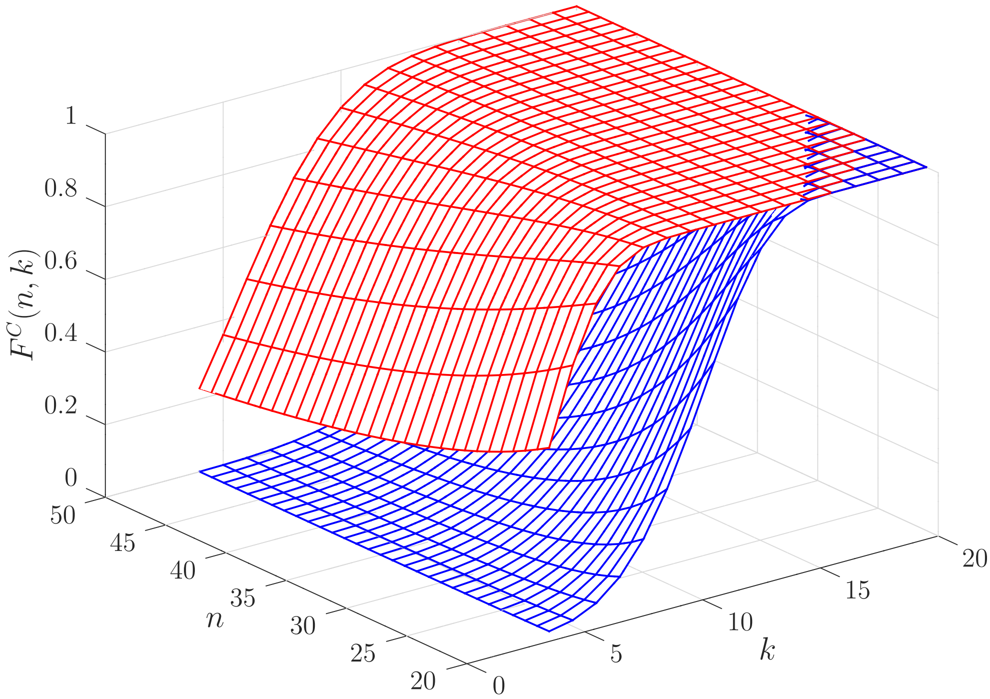

FIGURE 6.3: *Upper and lower bounds on the fixation probability of k cooperators in Region B of the ESS on a cycle of order n.*

smaller than the lower bound on the fixation probability of cooperation and so the strategy of cooperation is favoured in the ESS on a cycle in Region B of the (S,T)-phase plane above the strategy of defection.

Theorem 6.10 (An upper bound on the fixation probability of defection for $k \geq \frac{n}{4}$). *In the ESS on a cycle of order $n \geq 5$ with pay-off parameters S and T, satisfying $2S < T < S+1$, the fixation probability of a group of mutant defectors of size $k \geq n/4$ satisfies*

$$F_b^D(n,k) \leq \frac{\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{n-4i-1}{k-1}}{\binom{n-1}{n-k}}. \tag{6.16}$$

Proof: Consider placing $n-k$ cooperators at k vertex locations along a cycle of order n (one to the right of each defector) and identifying the number of ways in which this may be achieved such that each cooperation run has length at most 3. This number is also the number of non-negative integer solutions to the equation

$$x_1 + x_2 + x_3 + \cdots + x_k = n-k$$

in which each $x_1, \ldots, x_k \leq 3$. The latter number is the coefficient of the term z^{n-k} in the generating function

$$\left[z^0 + z^1 + z^2 + z^3\right]^k = (1-z^4)^k\left(\frac{1}{1-z}\right)^k.$$

This coefficient is

$$c_{n,k} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{n-4i-1}{k-1}.$$

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In order to obtain an upper bound on the desired fixation probability, this quantity $c_{n,k}$ is divided by the total number of placements of $n-k$ cooperators at k vertex locations along a cycle of order n with replacement, which is

$$\binom{n-1}{n-k}.$$

This quotient is an upper bound on the fixation probability $F_b^D(n,k)$ as there are instances which satisfy the aforementioned condition in which no cooperation run has length at least 4, and for which the fixation of defection is not guaranteed, such as, for example, instances containing the substate $CCDD$. $\square$

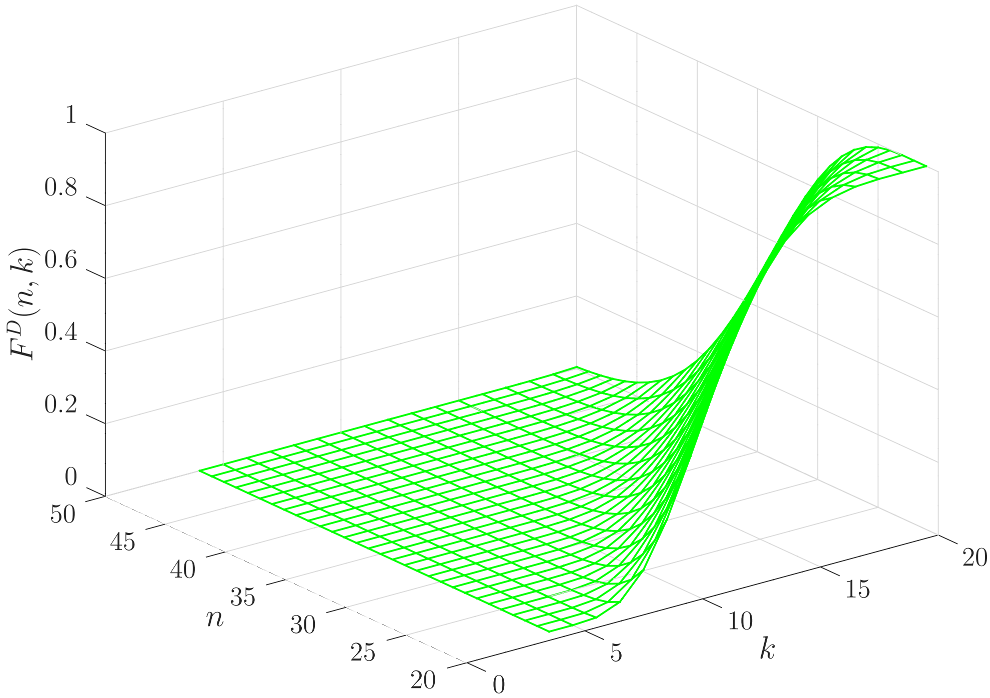

FIGURE 6.4: *Upper bound on the fixation probability of k defectors in Region B of the ESS on a cycle of order n.*

The upper bound of Theorem 6.10 on the fixation probability of k defectors is plotted in Figure 6.4 for $n \in \{20, \ldots, 50\}$ and for $k \in \{4, \ldots, 20\}$. It may be verified that for all values of n and k, the lower bound on the fixation probability of k mutant cooperators is greater than or equal to the upper bound of the fixation probability of k mutant defectors, and therefore, the strategy of cooperation is favoured in the ESS on a cycle of order n with pay-off parameter values satisfying $2S < T < S + 1$.

6.5 State graph component enumeration

A lower bound on the number of components in the state graph of the ESS on a cycle in Region B of the (S, T)-phase-plane is established in this section. Considering that in the steady states of the game, the only remaining defectors form part of oscillation clusters, the number of

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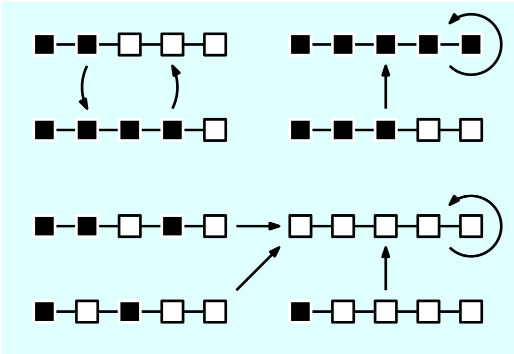
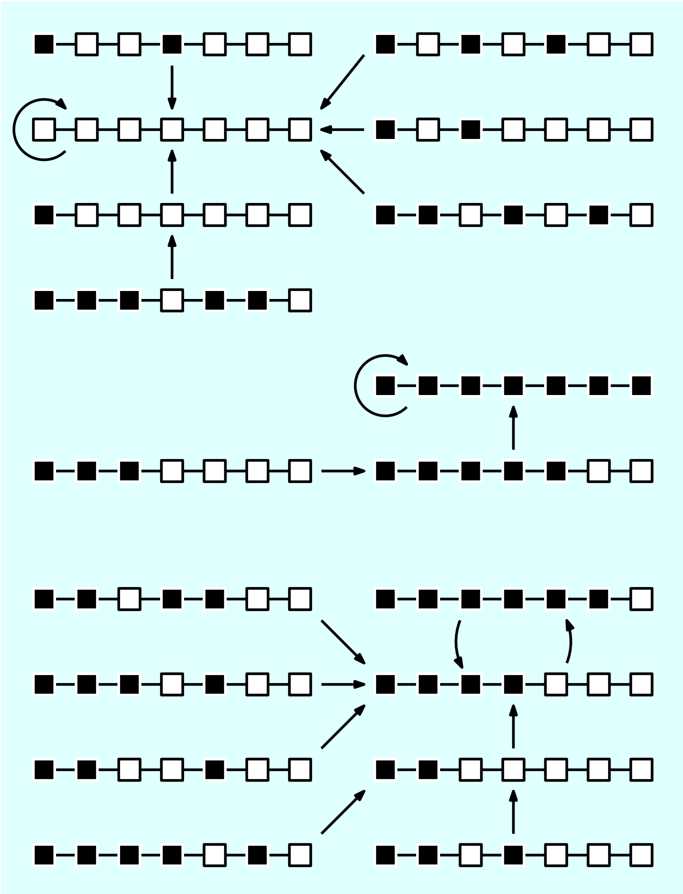
components in the state graph is bounded from below by the number of oscillation clusters that can fit into a cycle of order n and the ways in which these can be arranged. This is particularly tricky to enumerate because these oscillation clusters need not be synchronised, (there might, for example, be some of size 3 while others have size 1, and thus are out of phase). Consider, for reference, the state graphs for the ESS on cycles of orders $n = 5$ and $n = 7$ in Region B depicted in Figure 6.5(a) and 6.5(b), respectively.



(a) $n = 5$



(b) $n = 7$

FIGURE 6.5: *The state graphs for the ESS on a 5-cycle and a 7-cycle in Region B of the T-S phase plane in Figure 4.5. Filled squares denote players adopting the strategy of cooperation, while a blank squares denote players adopting the strategy of defection. Although players are represented in linear arrays, these arrays should be interpreted as wrapping around so that the first and last player in the array are adjacent.*

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Lemma 6.8 (Enumeration of in-phase limit cycles).
The number of components in the state graph of the ESS on a cycle of order n with pay-off parameter values satisfying $2S < T < S+1$ that consist of only in-phase oscillation clusters, is

$$\begin{aligned} Q_{in}^{b}(n) = \sum_{i=1}^{\lfloor \frac{n}{5} \rfloor} \frac{1}{2i} \Bigg\{ & \binom{n-4i-1}{i-1} + \sum_{j \in \mathcal{T}} \binom{(n-5i)d/i + d - 1}{d-1} \\ & + \frac{i}{2} \binom{\frac{n-4i}{2} - 1}{\frac{i}{2} - 1} (i+1) \pmod 2 \\ & + \sum_{m=0}^{\lfloor \frac{n-5i}{2} \rfloor} \Bigg[\frac{i}{2}(n-5i-2m+1) \binom{\frac{i-2}{2} + m - 1}{m} (i+1) \pmod 2 \\ & + i \binom{\frac{i-1}{2} + m - 1}{m} i \pmod 2 \Bigg] \Bigg\}, \end{aligned} \tag{6.17}$$

where $\mathcal{T} = \{x \in \mathbb{N} \mid i$ divides $n \gcd(i,x)$ and $x < i\}$ and $d = \gcd(i,j)$.

Proof: Let $\mathcal{Y}_i$ denote the number of limit cycles of the form

$$\underbrace{DDDCC\cdots}_{\text{structure 1}} \underbrace{DDDCC\cdots}_{\text{structure 2}} \cdots \underbrace{DDDCC\cdots}_{\text{structure } i} \tag{6.18}$$

This means that $\sum_{i=1}^{\lfloor \frac{n}{5} \rfloor} \mathcal{Y}_i$ is the total number of components in the state graph containing oscillation clusters in phase and therefore provides a lower bound on the number of components in the state graph. There are $n-5i$ cooperators that can be placed in the i structures indicated in (6.18) and this leads to an enumeration problem which can be solved by means of the Cauchy-Frobenius Lemma.

Let $\mathcal{Z}$ be the set of limit cycles of the form described above an let ι be the identity operator in the group $\mathcal{V}$ which acts on these states. Furthermore, let ρ^j be the permutation in which each structure is shifted j positions to the right in a modular fashion, and let δ be the action which reflects the structures about the diametrical axis passing through the first structure. The action δ^j depends on the parity of i. For odd i, the action δ^j reflects the structures about the diametrical axis passing through structure j. For even i, the action δ^j reflects the structures around the diametrical axis passing through structures j and $(j+i/2) \pmod i$ for $j = 1, \ldots, i/2$, while for $j = i/2, \ldots, i$ the action δ^j reflects the structures about the diametrical axis passing between structures $j+1 \pmod{i/2}$ and $j+2 \pmod{i/2}$. The group $\mathcal{V}$ is formed by the set $\{\iota, \rho^1, \rho^2, \ldots, \rho^{i-1}, \delta, \delta^2, \ldots, \delta^i\}$, which has order $2i$. According to the Cauchy-Frobenius Lemma,

$$\mathcal{Y}_i = \frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} |F_v|, \tag{6.19}$$

where F_v denotes the states in $\mathcal{Z}$ that are fixed by v.

Consider first the identity operator which maps each structure in (6.18) to itself and therefore fixes each element of $\mathcal{Z}$. In this case,

$$|F_\iota| = \binom{n-4i-1}{i-1}, \tag{6.20}$$

which is the number of ways of distributing the $n-5i$ cooperators among the i structures with replacement.

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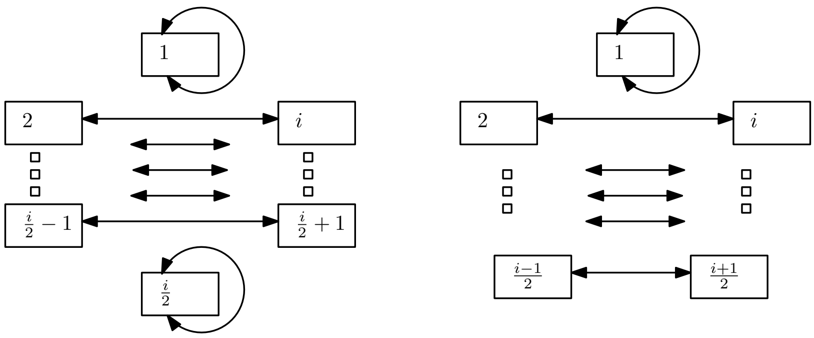
Shifting each structure j positions to the right by applying the action ρ^j fixes those states in which the first j structures determine the remaining $i-j$ structures, as long as $j|i$. Alternatively, the first $\gcd(i,j)$ structures determine the remaining structures, in the case where j does not divide i (which incidentally is j if $i|j$). Setting $d = \gcd(i,j)$, yields

$$|F_{\rho^j}| = \binom{(n-5i)d/i + d - 1}{d-1}.$$

That is, the number of ways of distributing the $(n-5i)d/i$ cooperators among the d structures. This can only occur if the number of cooperators is sufficiently large to continue the pattern i/d times, which is precisely when $i|(n\gcd(i,j))$.

The effect of the action δ is that the structures are swapped with one another in pairs, leaving either only the first structure in place if i is odd, or leaving both the first and middle (1^{st} and $\frac{i}{2}^{\text{th}}$) structures in place if i is even. This effect is illustrated in Figure 6.6.



(a) i *even* (b) i *odd*

FIGURE 6.6: *The effect of the group action δ on the positioning of structures labelled by number for both the cases where (a) i is even and (b) where i is odd.*

Thus, the number of states fixed by the action δ is the number of possible states in which each pair of switched structures is identical. For even i, distributing m cooperators amongst the first half of the structures that switch $\frac{i-2}{2}$ structures determines the placement of $2m$ cooperators in all of the structures, except for the two which are fixed to themselves. The remaining $n-5i-2m$ cooperators can then be distributed amongst the two remaining structures in

$$\binom{n-5i-2m+1}{1}$$

ways. In total that is

$$\binom{\frac{i-2}{2}+m-1}{m}(n-5i-2m+1)$$

arrangements for even i. For odd i, the distribution of m cooperators among the $\frac{i-1}{2}$ structures determines the placement of $2m$ cooperators amongst $i-1$ structures. The remaining cooperators must then be placed in the first structure, which can only be done in one way. This yields

$$\binom{\frac{i-1}{2}+m-1}{m}$$

arrangements for odd i. Adding these two quantities together and ensuring that the correct instances are added, by multiplication by $i-1 \pmod 2$ and $i \pmod 2$, respectively, yields the

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total number of arrangements

$$|F_\delta| = \sum_{m=0}^{\lfloor \frac{n-5i}{2} \rfloor} (n-5i-2m+1)\binom{\frac{i-2}{2}+m-1}{m}(i-1) \pmod 2 + \binom{\frac{i-1}{2}+m-1}{m} i \pmod 2. \quad (6.21)$$

For odd i and for even i (but only for $j = 1, \ldots, i/2$), the effect of the action δ^j is that the structures are reflected about the diametrical axis passing through the j^{th} structure. For odd i, this always fixes one structure in place and switches the remaining $\frac{i-1}{2}$ structures in pairs. For even i, on the other hand, two of the structures map to themselves and the remaining $\frac{i-2}{2}$ structures form pairs that switch. The number of ways in which this can be done for each j in question is the same as for δ.

For even i and for $j = i/2, \ldots, i$, the effect of the action δ^j is a reflection about the diametrical axis passing between two structures, starting between structures one and two, and ending between structures $i/2-1$ and $i/2$. This means that all the structures form pairs which map to one another.

Thus, for even i, the number of arrangements fixed by δ^j (for $j = 1, \ldots, i/2$) is given by

$$\binom{\frac{i}{2} + \frac{n-5i}{2} - 1}{\frac{n-5i}{2}} = \binom{\frac{n-4i}{2} - 1}{\frac{i}{2} - 1},$$

which is the number of possible distributions of $\frac{n-5i}{2}$ cooperators among $\frac{i}{2}$ structures with replacement.

Therefore,

$$\begin{aligned}\sum_{j=1}^{i} |F_{\delta^j}| = &\frac{i}{2}\binom{\frac{n-4i}{2}-1}{\frac{i}{2}-1}(i+1) \pmod 2 \\ &+ \sum_{m=0}^{\lfloor \frac{n-5i}{2} \rfloor} \Bigg[\frac{i}{2}(n-5i-2m+1)\binom{\frac{i-2}{2}+m-1}{m}(i+1) \pmod 2 \\ &+ i\binom{\frac{i-1}{2}+m-1}{m} i \pmod 2\Bigg].\end{aligned}$$

□

The number of limit cycles of the form (6.18) is enumerated in Table 6.3 using Lemma 6.8 for $n \in \{5, \ldots, 38\}$.

TABLE 6.3: $Q^b_{in}(n)$, *the number of limit cycles of the ESS on a cycle of order* $n \in \{5, \ldots, 38\}$ *in Region B of the* (S, T)*-phase plane in which each oscillation cluster is in-phase.*

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$Q^b_{in}(n)$	1	1	1	1	1	2	2	3	3	4	5	6	7	9	10	13	15	19

23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38
22	29	34	43	52	66	80	103	125	160	198	253	316	406	508	653

Limit cycles may, however, also assume the form

$$\underbrace{\overbrace{DDDCC\cdots}^{\text{box 1}}\overbrace{DCC\cdots}^{\text{box 2}}}_{\text{structure 1}}\ \underbrace{\overbrace{DDDCC\cdots}^{\text{box 3}}\overbrace{DCC\cdots}^{\text{box 4}}}_{\text{structure 2}}\ \cdots\ \underbrace{\overbrace{DDDCC\cdots}^{\text{box } 2i-1}\overbrace{DCC\cdots}^{\text{box } 2i}}_{\text{structure } i}, \quad (6.22)$$

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which can further be partitioned into runs within each box, such as

$$\overbrace{\underbrace{DDD}_{\text{run }1}\underbrace{CC\ldots}_{\text{run }2}}^{\text{box }1}\overbrace{\underbrace{D}_{\text{run }3}\underbrace{CC\cdots}_{\text{run }4}}^{\text{box }2}\cdots\overbrace{\underbrace{DDD}_{\text{run }4i-4}\underbrace{CC\cdots}_{\text{run }4i-3}}^{\text{box }2i-1}\overbrace{\underbrace{D}_{\text{run }4i-1}\underbrace{CC\cdots}_{\text{run }4i}}^{\text{box }2i}. \tag{6.23}$$

The form (6.22) represents those limit cycles in which the oscillation clusters are all of alternating phase. This is the densest packing of oscillation clusters possible, resulting from the shared cooperator/defector between two oscillation clusters. Note that in each structure there are two possible *boxes* into which cooperators may be placed (a cooperator placed in box j is placed specifically in *run* $2j$ and thus specifying the box is sufficient in order to know exactly where the cooperator is placed). Furthermore, it is clear that counting these limit cycles results in no overlap with those described in Lemma 6.8, as these are completely alternating in phase.

Lemma 6.9 (Enumeration of alternating phase limit cycles).
The number of components in the state graph of the ESS on a cycle of order n with pay-off parameter values satisfying $2S < T < S+1$, that consist of only alternating phase oscillation clusters, is

$$\begin{aligned} Q^b_{alt}(n) = \sum_{i=1}^{\lfloor \frac{n}{8} \rfloor} \frac{1}{2i} \Bigg[\binom{n-6i-1}{2i-1} + \sum_{j\in\mathcal{K}} \binom{\frac{n-6i}{i}d-1}{2d-1} \\ + i\binom{\frac{n-6i}{2}-1}{i-1}(n+1) \pmod 2 \Bigg], \end{aligned} \tag{6.24}$$

where $\mathcal{K} = \{x \in \mathbb{N} \mid i \text{ divides } n\gcd(i,x) \text{ and } x < i\}$.

Proof: It is evident from (6.22) that even boxes map to even boxes for fixation under an action, and that the odd boxes, similarly, map to odd boxes for the arrangement to be fixed under that action. In (6.22), the number of determined symbols is given by $8i$, and the remaining $n-8i$ symbols have to be distributed among $2i$ distinguishable boxes ($2i$ distinguishable cooperation runs). Let $\mathcal{U}$ be the total number of limit cycles of the form (6.22). The action ι is the identity which maps each run to itself. The action ρ^j shifts each run $4j$ positions to the right in a modular fashion. Lastly, the action σ_k reflects the runs around the diametrical axis passing through the defection runs of boxes k and $k+i$ (runs $2k-1$ and $2(k+i)-1$). Then the group of actions $\mathcal{G}$ consists of the set $\{\iota, \rho^1, \rho^2, \ldots, \rho^{i-1}, \sigma_1, \sigma_2, \ldots, \sigma_i\}$. According to the Cauchy-Frobenius Lemma,

$$\mathcal{U} = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} |F_g|,$$

where F_g is the number of arrangements of the $n-8i$ symbols that remain fixed under the action g. Every arrangement is fixed under the action ι as this action maps each box to itself. Thus,

$$|F_\iota| = \binom{2i+n-8i-1}{n-8i} = \binom{n-6i-1}{2i-1}. \tag{6.25}$$

The action ρ^j fixes those arrangements in which the first $2\gcd(i,j)$ boxes are free and are then repeated $2i/2\gcd(i,j)$ times. Setting $d = \gcd(i,j)$ means that distributing $\frac{n-8i}{2i}2d$ symbols among $2d$ boxes determines the remaining symbols' placements, as long as i divides $n\gcd(i,x)$. Therefore,

$$|F_{\rho^j}| = \binom{2d+\frac{n-8i}{i}d-1}{2d-1} = \binom{\frac{n-6i}{i}d-1}{2d-1}. \tag{6.26}$$

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Lastly, the action σ_k maps exactly two defection runs onto themselves and the remaining $4i-2$ runs are switched in pairs, consistently mapping defection runs of the same length onto one another. Assigning $(n-8i)/2$ cooperators to the i cooperation runs from run $2k$ to $2(k+i)$ determines the distribution of all $n-8i$ cooperators among all $2i$ cooperation runs. Therefore, the number of arrangements fixed by σ_k is given by

$$\binom{i+\frac{n-8i}{2}-1}{\frac{n-8i}{2}} = \binom{\frac{n-6i}{2}-1}{i-1}. \tag{6.27}$$

This is, however, only the case if $n-8i$ is even, which occurs when n is even, and so multiplying by $n+1 \pmod 2$ ensures that the quantity in (6.27) is only counted when n is even. Therefore,

$$\sum_{k=1}^{i} |F_{\sigma_k}| = i\binom{\frac{n-6i}{2}-1}{i-1}(n+1) \pmod 2. \qquad \square$$

The number $Q^b_{alt}(n)$ is tabulated in Table 6.4 for $n \in \{8, \dots, 38\}$.

TABLE 6.4: *$Q^b_{alt}(n)$, the number of limit cycles of the ESS on a cycle of order $n \in \{8, \dots, 38\}$ in Region B of the (S,T)-phase plane in which each oscillation cluster is in alternating phase.*

n	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$Q^b_{alt}(n)$	1	1	2	2	3	3	4	4	6	6	10	11	18	21	32	38	55

25	26	27	28	29	30	31	32	33	34	35	36	37	38
65	91	111	154	193	271	348	485	634	872	1 142	1 562	2 048	2 783

The two lemmas above provide the numbers of components in the state graph of the ESS on a cycle of order n consisting either of only in-phase oscillation clusters, or else of completely alternating phase oscillation clusters, giving rise to the following theorem.

Theorem 6.11 (A lower bound on the number of components in the state graph). *The number $Q^b(n)$ of components in the state graph of the ESS on a cycle of order n with pay-off parameters S and T, satisfying $2S < T < S+1$, satisfies*

$$Q^b(n) \geq Q^b_{in}(n) + Q^b_{alt}(n) + 2. \tag{6.28}$$

Proof: There is no overlap between the set of limit cycles containing alternating-phase oscillation clusters and the set of in-phase oscillation clusters, and so their cardinalities may be summed. Furthermore, the addition of the constant 2 in (6.28) may be accounted for by the all-cooperator and all-defector steady states, each in a component of their own. There, however, exist limit cycles in which the pattern of oscillation clusters is not completely in-phase or completely alternating in phase, such as a steady state of the form

$$DDDCC\cdots\ DDDCC\cdots\ DCC\cdots\ \cdots\ DDDCC\cdots, \tag{6.29}$$

which is clearly not counted in either of the cardinalities $Q^b_{in}(n)$ or $Q^b_{alt}(n)$. Therefore, the right hand side of (6.28) is only a lower bound on the number of state graph components. $\square$

The lower bound in Theorem 6.11 on the number of components in the state graph of the ESS on cycle of order n is tabulated in Table 6.5 and illustrated graphically in the Figures 6.7 and 6.8 on linear and log-linear axes formats, respectively.

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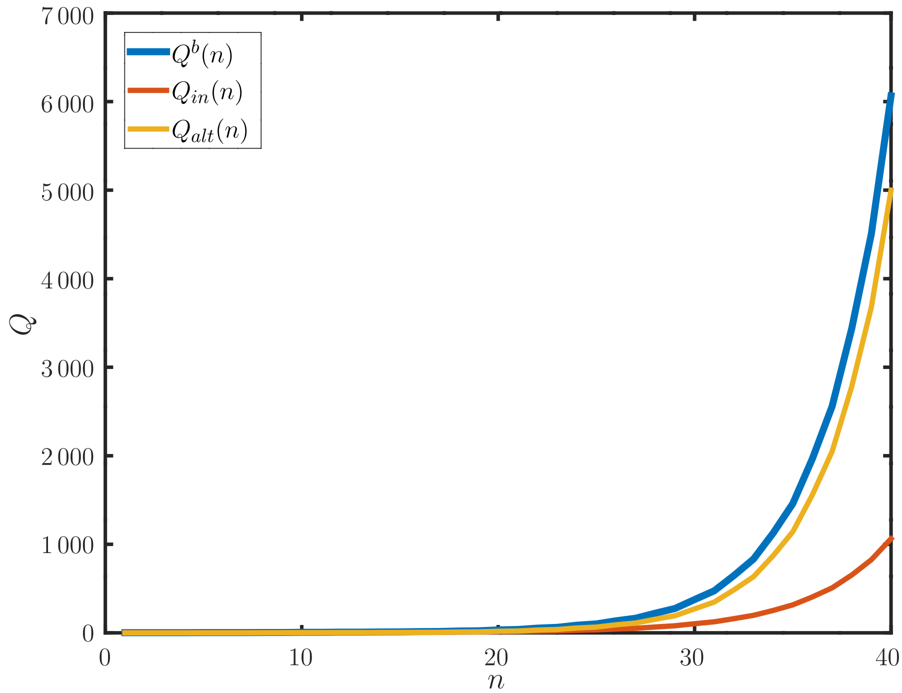

FIGURE 6.7: *The lower bound of Theorem 6.11 on the number of components in the state graph of the ESS on a cycle of order n in Region B of the (S, T)-phase plane on a linear vertical axis.*

TABLE 6.5: *$Q^b(n)$, the lower bound of Theorem 6.11 on the number of components in the state graph of the ESS on a cycle of order $n \in \{1, \ldots, 38\}$ in Region B of the (S, T)-phase plane.*

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$Q^b(n)$	2	2	2	2	3	3	3	4	4	6	6	8	8	10

15	16	17	18	19	20	21	22	23	24	25	26	27
11	14	15	21	23	33	38	53	62	86	101	136	165

28	29	30	31	32	33	34	35	36	37	38
222	275	376	475	647	834	1 127	1 460	1 969	2 558	3 438

6.6 Chapter summary

This chapter opened in §6.1 with the establishment of a number of fundamental results used throughout the remainder of the chapter. These results were used in §6.2 to determine which initial states necessarily lead to persistent cooperation. These initial states are characterised by the presence of either of the substates $CCDD$ and/or $CCCC$ in the initial state. In §6.3, the transfer matrix method was applied in order to determine the probability of persistent cooperation, given a random assignment of strategies to players on cycles of order n. Upper and lower bounds on the fixation probability of the strategy of cooperation were presented in §6.4 for the ESS in Region B of the (S, T)-phase plane, along with an upper bound on the fixation probability of the strategy of defection for the ESS in the same region of the phase plane. This

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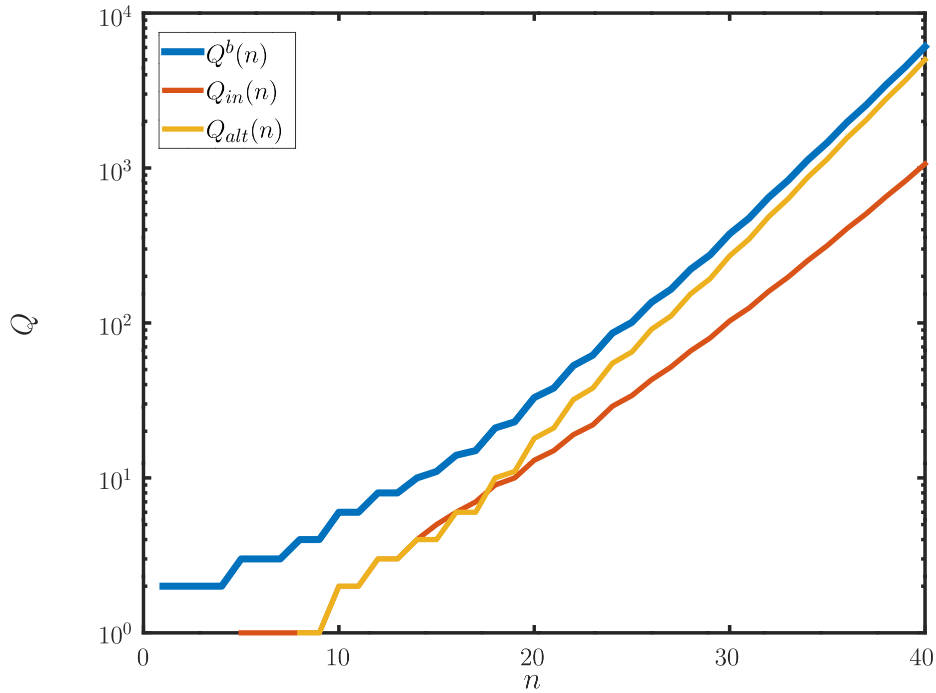

FIGURE 6.8: *The lower bound of Theorem 6.11 on the number of components in the state graph of the ESS on a cycle of order n in Region B of the (S,T)-phase plane on log-linear axes.*

result was used to show that in Region B of (S,T)-phase plane, the strategy of cooperation is favoured above that of defection in the ESS on a cycle. Lower bounds on the number of components in the state graph of the ESS on a cycle in Region B of the (S,T)-phase plane were finally presented in §6.5, making use of the Cauchy-Frobenius Lemma to enumerate orbits of limit cycles in which each oscillation cluster is either in-phase or alternating in phase.

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CHAPTER 7

The case where $T < 2S < S+1$ (Region C)

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The long-time asymptotic behaviour of the ESS on a cycle of order n in Region C of the (S,T)-phase plane in Figure 4.5 is analysed in this chapter. The chapter opens in §7.1 with a characterisation of initial states that lead to some form of persistent cooperation. The probability that a randomly generated initial state leads to some form of persistent cooperation is derived in §7.2 by leveraging the aforementioned characterisation. A variation on the notion of fixation probability is considered in §7.3 with an emphasis on comparing this notion for the strategies of cooperation and defection. The chapter closes in §7.4 with an enumeration of the components in the state graph of the ESS on a cycle in Region C of the (S,T)-phase plane and a summary of its contents in §7.5.

The results established in this chapter are specific to situations in which the inequality chain

$$0 < \underline{T < 2S < S+1} < 2 < 2T \tag{7.1}$$

hold, which is the case in Region C of the (S,T)-phase plane. As a point of departure, consider the ESS state graphs for cycles of order $n = 5$ and $n = 7$ in Figures 7.1(a) and 7.1(b), respectively. These graphical representations shed light on some of the dynamics present in the ESS on a cycle in Region C of the (S,T)-phase plane. As a first observation, notice the presence of a limit cycle in the state graph for a cycle of order $n = 5$ in Figure 7.1(a) and one state leading into this limit cycle. Secondly, notice that the states leading to some form of persistent cooperation outnumber those leading to the all-defector states in both the cases when $n = 5$ and when $n = 7$. A *transient state* is one in which the players are indeed changing their strategies from one round to the next, yet the resulting state is isomorphic to the original state. As final preliminary observation, notice that there is a *transient state* in the state graph for the case when $n = 7$, namely the state $CCDCDDD$. This state certainly has players changing their

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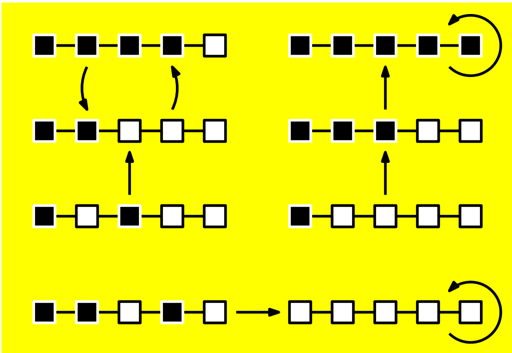
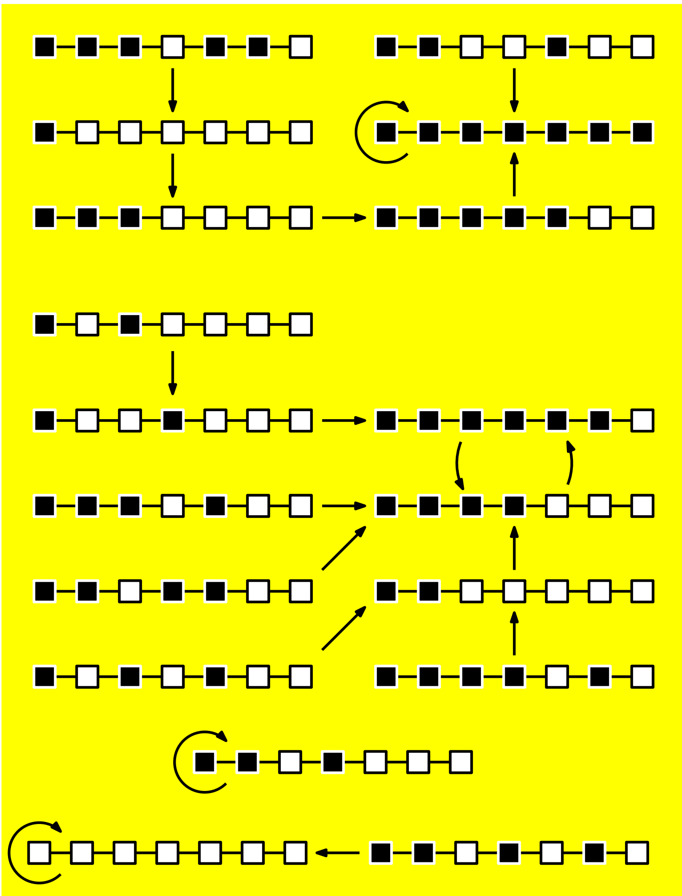
strategies from one game round to the next, as will be described in the chapter, yet the resulting state to which it transitions, $CDDDCDC$, is isomorphic to the original state.



(a) $n = 5$



(b) $n = 7$

FIGURE 7.1: *State graphs for the ESS on a 5-cycle and a 7-cycle in Region C of the T-S phase plane in Figure 4.5. A solid square denotes a player adopting the strategy of cooperation, while an open square denotes a player adopting the strategy of defection. Although players are represented in linear arrays, these arrays should be interpreted as wrapping around so that the first and last player in the array are adjacent.*

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7.1 Initial states leading to persistent cooperation

The requirements for persistent cooperation within the ESS on a cycle in Region C of the (S, T)-phase plane are established in this section by means of a characterisation of initial states that lead to persistent cooperation. The first result describes the requirements for strategy retention of a defector adjacent to at least one cooperator. A consequence of this result is that even a singleton cooperator may retain its strategy during the subsequent game round if the defectors to which it is adjacent are not themselves singletons.

Lemma 7.1 (Singleton defectors retain their strategy).
In the ESS on a cycle of order n and with pay-off parameters S and T satisfying $T < 2S < S+1$, a defector adjacent to a cooperator retains its strategy during the following game round if and only if it is flanked by cooperators. The cooperators adjacent to a singleton defector adopt the strategy of defection during the following game round.

Proof: By contradiction. Consider the central defector x in the substate CD_xD and suppose, to the contrary, that it retains its strategy during the following game round. This means that the pay-off value obtained by player x is at least as large as that of the cooperator to which it is adjacent. The pay-off value obtained by the cooperator is at least $2S$, however, while the pay-off value obtained by player x is at most $T < 2S$, a contradiction.

Now consider the substate CD_xC, in which the central defector x is adjacent to two cooperators. The defector obtains a pay-off value of $2T$, which is indeed the largest pay-off value achievable in the game and so x will certainly retain its strategy of defection during the following game round. Furthermore, the adjacent cooperators obtain a pay-off value of at most $S+1 < 2T$ each and so adopt the strategy of defection during the following game round. □

The next result establishes the minimum length of a cooperation run required for persistent cooperation which, in the case of Region C of the (S, T)-phase plane, is three.

Lemma 7.2 (A sufficient condition for persistent cooperation).
In the ESS on a cycle of order $n \geq 4$ and with pay-off parameters S and T satisfying $T < 2S < S + 1$, a cooperation run of length at least 3 leads to persistent cooperation.

Proof: Consider a cooperation run of length at least 3. This cooperation run is flanked by either:

1. a defection run of length at least 2 on each side (possibly the same one if $n = 5$),
2. one defection run of length precisely 1 and one of length at least 2, or
3. a defection run of length precisely 1 on each side (possibly the same one if $n = 4$).

In cases 1 and 2 above, the defection runs of length at least 2 will diminish in length, by Lemma 7.1, round after round as long as they have length at least 2. In case 3 above, the singleton defectors adjacent to the cooperation run become defection runs of length 3 during game round $t + 1$ as each defector obtains a pay-off value of $2T$ while the cooperators each obtains pay-off values of at most $S + 1 < 2T$. Therefore, during game round $t + 1$, the original cooperation run contains at least one cooperator adjacent to defection runs of length at least 3 on each side. By Lemma 7.1, these defection runs decrease in length during the subsequent game rounds as long as they have length at least 2. □

The results of the two lemmas presented thus far are synthesised into a theorem explicating the necessary and sufficient substates leading to persistent cooperation in Region C of the (S, T)-phase plane.

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Theorem 7.1 (Characterisation of initial states leading to persistent cooperation). *In the ESS on a cycle of order $n \geq 3$ and with pay-off parameters S and T satisfying $T < 2S < S + 1$, cooperation persists if and only if the initial state contains one or both of the substates CDD and/or CCC.*

Proof: Cooperation persists for any initial state containing the substate CCC by Lemma 7.2.

In any initial state consisting of alternating strategies $CDCDCD\cdots$, each cooperator obtains a pay-off value of $2S$ while each defector obtains pay-off value of $2T > 2S$. Therefore, each cooperator resorts to the strategy of defection during the subsequent game round. Initial states not consisting solely of alternating strategies contain either one or both of the substates CDD and/or CCD.

By Lemma 7.1, the substate CDD leads to persistent cooperation as each defection run of length at least 2 will diminish in length as long as it has at least one cooperator on either side, while the adjacent cooperation runs grow in length and so cooperation will persist.

The only larger substate containing CCD, that does not include either of the substates CDD or CCC, both of which yield persistent cooperation as argued above, is $CD\boldsymbol{CCD}C$. In this substate, each of the defectors obtains the pay-off value $2T$ while each of the cooperators obtains a pay-off value of $S + 1 < 2T$, and so the cooperators adopt the strategy of defection during the following game round. □

A sufficient condition for persistent cooperation in Region C of the (S, T)-phase plane is therefore the presence of either of the states CCC and/or CDD in the initial state of the game, and there are no other substates that can lead to persistent cooperation.

7.2 Probability of persistent cooperation

This section is devoted to establishing the probability of persistent cooperation from a randomly initialised game state in the Region of the (S, T)-phase plane where $T < 2S < S + 1$. The transfer-matrix method is again used to establish a recurrence relation which yields the number of possible initial states that do not lead to persistent cooperation (excluding the all-defector state). Dividing the number of initial states that do not lead to persistent cooperation by the total number of possible initial states (2^n) yields the probability of no persistent cooperation and this can, in turn, can be used to determine the probability of persistent cooperation.

Let each of the substates mentioned in Theorem 7.1 be considered *forbidden strings.* Then the digraph D_2, shown in Figure 7.2, contains four vertices, each representing a possible binary string of length 2. In this digraph, each vertex v_i, representing the string s_1s_2, is incident to each other vertex v_j, representing the string s_2s_3, if and only if the string $s_1s_2s_3$ contains no forbidden strings.

The adjacency matrix of D_2 is given by

$$\boldsymbol{C} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}. \tag{7.2}$$

But considering that D_2 has two components, one of which will always include only the all-defector state, this graph can be simplified by considering only vertices v_1, v_2 and v_3. The

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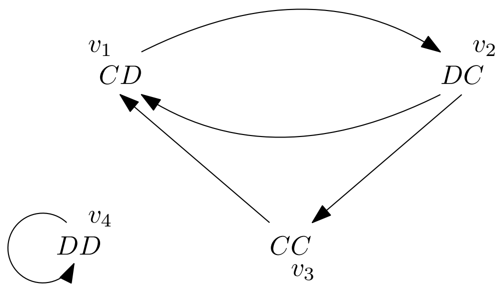

FIGURE 7.2: *A digraph D_2 used to calculate the number of strings of length n that do not contain the strings mentioned in Theorem 7.1.*

adjacency matrix of this modified digraph, denoted by D_2^*, is

$$\boldsymbol{C^*} = \begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}. \tag{7.3}$$

Now $\det(\boldsymbol{I} - x\boldsymbol{C^*}) = 1 - x^2 - x^3$, where $\boldsymbol{I}$ is the 3×3 identity matrix. Let c_n denote the number of initial states (other than the all-defector state) for the ESS on a cycle of order n that do not lead to persistent cooperation. Then

$$\sum_{n=1}^{\infty} c_n x^n = \frac{-x(-2x - 3x^2)}{1 - x^2 - x^3} \tag{7.4}$$

is a generating function for the sequence c_1, c_2, c_3, ... Furthermore, the recurrence equation

$$c_n = c_{n-2} + c_{n-3} \tag{7.5}$$

may be used to calculate the value of c_n for all $n \geq 4$. The seed values for this recurrence equation may be determined by the Maclaurin series expansion

$$\frac{-x(-2x - 3x^2)}{1 - x^2 - x^3} = 2x^2 + 3x^3 + 2x^4 + \dots \tag{7.6}$$

of (7.4). The seed values to the recurrence equation (7.5) are therefore $c_1^* = 0$, $c_2^* = 2$ and $c_3^* = 3$. Note, that the actual number of initial states that do not lead to persistent cooperation is given by $c_n + 1$, including the all-defector state. Having established the number of possible initial states that do not allow for persistent cooperation, it follows that the number of states that do indeed allow for persistent cooperation is the complement $2^n - c_n - 1$. Thus the probability of persistent cooperation resulting from a randomly generated assignment of strategies on a cycle of order n within the context of the ESS in Region C (where $T < 2S < S + 1$) is

$$P_c(n) = 1 - \frac{(c_n + 1)}{2^n}. \tag{7.7}$$

Intuitively, a sequence defined as the sum of two previous, non-negative terms in the sequence is expected to be increasing, yet it can be seen in Table 7.1 that the fourth term of the sequence decreases from the third. It can also be seen that the sixth term is equal to the fifth. In the following lemma it is shown that the sequence strictly increases from the sixth term onward. The probability of persistent cooperation in the ESS on a cycle of order n in Region C of the (S, T)-phase plane is plotted in Figure 7.3 as a function of n.

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TABLE 7.1: *Values of c_n and 2^n for $n \in \{1, \dots, 10\}$ used to compute the probability $P_c(n) = 1-(c_n+1)/2^n$ of persistent cooperation resulting from a randomly generated initial state in the ESS on a cycle of order n, with pay-off parameter values satisfying $T < 2S < S + 1$.*

$n \to$	1	2	3	4	5	6	7	8	9	10
$c_n + 1$	1	3	4	2	5	5	7	10	12	17
2^n	2	4	8	16	32	64	128	256	512	1 024

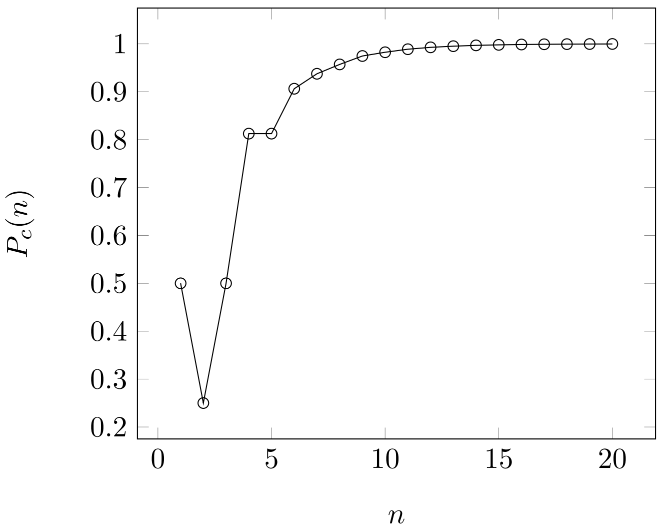

FIGURE 7.3: *The probability $P_c(n) = 1 - c_n/2^n$ of persistent cooperation in the ESS on a cycle of order n in Region C of the (S, T)-phase plane as a function of n.*

Lemma 7.3.
The sub-sequence $c_6, c_7, c_8, \dots$ satisfying (7.5), with seed values $c_1^* = 0$, $c_2^* = 2$ and $c_3^* = 3$, is strictly increasing.

Proof: By induction over n. Observe as base case for the induction process that $c_7 > c_6 > 0$ in Table 7.1. Assume, as induction hypothesis, that $c_n > c_{n-1} > 0$ for all $n \leq k$. Finally, observe as induction step that

$$\begin{aligned} c_{k+1} - c_k &= (c_{k-1} + c_{k-2}) - (c_{k-2} + c_{k-3}) \\ &= c_{k-1} - c_{k-3} > 0 \end{aligned}$$

because $c_{k-1} > c_{k-2} > c_{k-3}$ by the induction assumption. □

The fact that the sequence $P_c(1)$, $P_c(2)$, $P_c(3), \dots$ is increasing for all $n \geq 6$ may be leveraged to show that the limit of $P_c(n)$ as $n \to \infty$ is unity.

Theorem 7.2 (Probability of persistent cooperation in the asymptotic of n).
The probability $P_c(n)$ that a randomly generated initial state of the ESS on a cycle of order n with pay-off parameter values satisfying $T < 2S < S + 1$ results in some form of persistent cooperation satisfies

$$\lim_{n\to\infty} P_c(n) = \lim_{n\to\infty} \left(1 - \frac{c_n + 1}{2^n}\right) = 1.$$

Proof: Setting $J_n = c_{n-3}$ yields $J_n = c_{n-5} + c_{n-6}$ and $c_n = c_{n-2} + J_n$ by (7.5). Therefore,

$$c_{n-2} - J_n = c_{n-4} + c_{n-5} - (c_{n-5} + c_{n-6}), \tag{7.8}$$

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which simplifies to

$$c_{n-2} - J_n = c_{n-4} - c_{n-6}.$$

It follows from Lemma 7.3 that $c_{n-4} - c_{n-6} > 0$ and hence that $c_{n-2} > J_n$. This means that $0 < c_n < 2c_{n-2}$ and so, by Lemma 7.3, $2c_{n-1} > 2c_{n-2}$. Therefore, $0 < c_n < 2c_{n-1}$. Dividing the latter inequality chain right through by 2^n yields

$$0 < \frac{c_n}{2^n} < \frac{c_{n-1}}{2^{n-1}},$$

from which it follows that the sequence $\frac{c_6}{2^6}, \frac{c_7}{2^7}, \frac{c_8}{2^8}, \ldots$ remains positive and is strictly decreasing. The Monotonic Sequence Theorem [32] therefore guarantees convergence of the sequence.

Having established that the sequence converges, denote its limiting value by

$$\lim_{n\to\infty} \frac{c_n}{2^n} = W. \tag{7.9}$$

Then it follows from (7.5) that

$$\begin{aligned}
W &= \lim_{n\to\infty} \frac{c_{n-2} + c_{n-3}}{2^n} \\
&= \lim_{n\to\infty} \frac{c_{n-2}}{2^n} + \lim_{n\to\infty} \frac{c_{n-3}}{2^n} \\
&= \frac{1}{2^2} \lim_{n\to\infty} \frac{c_{n-2}}{2^{n-2}} + \frac{1}{2^3} \lim_{n\to\infty} \frac{c_{n-3}}{2^{n-3}} \\
&= \left(\frac{1}{2^2} + \frac{1}{2^3}\right) \lim_{n\to\infty} \frac{c_n}{2^n} \\
&= \left(\frac{1}{2^2} + \frac{1}{2^3}\right) \lim_{n\to\infty} W,
\end{aligned}$$

which implies that $W = 0$ and, consequently, that $\lim_{n\to\infty} P_c(n) = 1 - 0 = 1$. □

7.3 Fixation probabilities

In this section, a variation on the notion of fixation probability for the strategies of cooperation and defection in the ESS on a cycle with pay-off parameter values satisfying $T < 2S < S + 1$, are investigated. The section opens with a general investigation into the long-term asymptotic behaviour of the game. Bounds on the fixation probabilities are subsequently established in order to facilitate a comparison between the fixation probabilities of the two strategies.

7.3.1 Long-term behaviour of the game

The following lemmas describe the long-term behaviour of the game in Region C and are required for a meaningful analysis into the fixation probabilities of the strategies of cooperation and defection later on.

Lemma 7.4 (The nature of oscillation clusters).
In the ESS on a cycle of order n with pay-off parameters S and T, satisfying $T < 2S < S + 1$, defection runs of length at least 3 shrink by 2 in length during each game round until they disappear or reach length 1 and form an oscillation cluster which alternates between having length 1 and length 3.

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Proof: By definition each defection run is flanked by cooperators on either side. Defection runs of length at least 3 thus contain instances of the substate CDD. The defectors adjacent to the cooperator in such instances adopt the strategy of cooperation during the following game round by Lemma 7.1. Defection runs of length at least 3 therefore shrink by 2 in length during each game round until they either disappear (if their lengths are even) or until they have length 1 (if their lengths are odd).

The remaining defection runs of length 1 during some game round t are each flanked by at least two cooperators on each side, as one cooperator on each side was required to shrink from length 3 to length 1 and one cooperator was introduced on each side during the transition of the run from length 3 to 1. The singular defector receives the largest pay-off value achievable in the game and subsequently, by Lemma 7.1, that defection run of length 1 grows to a run of length 3 during game round $t + 1$. Again by Lemma 7.1, this substate $CDDDC$ reduces to a defection run of length 1 during game round $t + 2$. This set of three alternating players is called an oscillation cluster and will continue to alternate between a defection run of length 1 and one of length 3. □

The result of Lemma 7.4 is useful in describing the nature of limit cycles and steady states reached from initial states containing the substates CDD and/or CCC, which is the remit of the following lemma.

Lemma 7.5 (Long-term behaviour of the game).
In the ESS on a cycle of order n with pay-off parameters S and T, satisfying $T < 2S < S + 1$, an initial state containing the states CCC and/or CDD leads to either the all-cooperator steady state or to a limit cycle in which the remaining defection runs are all part of oscillation clusters.

Proof: Any part of an initial state in which the substate CCC is not formed as part of a larger cooperation run or in which the substate CDD is not formed is either a large defection run, the ends of which contain instances of the substate CDD or comprises instances (possibly overlapping) of the substate CDC.

In any instance of the substate CDC during game round 1, the defector is a singleton and thus, by Lemma 7.1, retains its strategy while the adjacent cooperators adopt the strategy of defection during the following game round, resulting in a defection run of length at least 3 flanked by a cooperator on either side[1]. During game round 2, the game state is such that all defection runs have length at least 3 (except those that had length 3 (or 4) during the first game round and have therefore already shrunk to length 1 (or 2)) and there exists at least one cooperation run.

From game round 2 onwards, each defection run is either already contained in an oscillation cluster or shrinks by 2 in length during each game round until it disappears or forms part of an oscillation cluster, by Lemma 7.4. If all defection runs disappear, the game enters the all-cooperator steady state while if there is at least one oscillation cluster, it enters a limit cycle defined by the number, position and phase of the oscillation clusters. □

The requirements for fixation described in §4.5 are twofold, involving both establishment and growth. Furthermore, as in the analysis of §6.4, the fixation probability of the strategy of cooperation is taken to be the probability of persistent cooperation in the context of k cooperators and $n - k$ defectors, for some $k \in \mathbb{N}$, distributed randomly along the cycle in the initial game round, while the fixation probability of the strategy of defection is taken as the probability of no persistent cooperation in the context of k defectors and $n - k$ cooperators, for some $k \in \mathbb{N}$, distributed randomly along the cycle in the initial game round. This is due to the game behaviour

[1]This cooperator was provided by the nearest instance of either the substate CCC or the substate CDD. The worst-case scenario features a singleton cooperator remaining amongst one large defection run to which the cooperator is adjacent on both sides.

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described in Theorem 7.5 from which it is clear that the trend is for cooperation to grow if it persists and for defection to shrink if cooperation persists.

7.3.2 The fixation probability of the strategy of cooperation

The focus now shifts to the fixation probability of cooperation. The nature of persistent cooperation, as described in Lemmas 7.4 and 7.5, shows that short runs of cooperation can grow to be larger in length while runs of defection decline in length until there is only a singular permanent defector or none at all. Therefore the fixation probability of the strategy of cooperation is the probability of persistent cooperation in the ESS on a cycle with k cooperators and $n-k$ defectors, for some $k \in \mathbb{N}$, distributed randomly along the cycle in the initial game round.

A singular cooperator introduced into a population of defectors on a cycle of order $n \geq 3$ will certainly perpetuate the growth of cooperation by Lemma 7.1. Furthermore, the strategy of cooperation will grow to cover the majority of the cycle, with the possible exception of one oscillation cluster, by Lemma 7.5. The fixation probability of a singleton cooperator is formalised in the following lemma.

Lemma 7.6 (Fixation probability of a singular cooperator).
The fixation probability of a singleton cooperator among a population of defectors in the pay-off parameter region of the ESS satisfying $T < 2S < S + 1$ on cycle of order $n \geq 3$ is 1.

Proof: Any singleton cooperator is flanked on both sides by a substate of the form CDD and so the large defection run will shrink in length during each game round until it disappears or forms an oscillation cluster by Lemma 7.5. Therefore the number of permanent defectors is at most one while the number of cooperators is at least $n-1$. □

In order to avoid the substates CCC, and more importantly CDD, each defector has to be a singleton, and this requires at least $n/2$ cooperators. This means that until the length of the group of entering mutant cooperators is $n/2$, the fixation probability remains 1. This observation is formalised in the following theorem.

Theorem 7.3 (Fixation probability of the cooperation strategy).
In the ESS on a cycle of order n with pay-off parameters S and T, satisfying $T < 2S < S + 1$, the fixation probability $F_c^C(n,k)$ of an entering group of k mutant cooperators is

$$F_c^C(n,k) = 1, \tag{7.10}$$

for all $k < n/2$.

Proof: Arranging the players along the cycle may be thought of as placing the k cooperators and considering each of them to be holding a container on its right-hand side. The $n-k$ defectors then have to be distributed among the k containers. For $k < n/2$, it follows that $n-k > k$ and so by the pigeonhole principle, there is at least one container into which at least two defectors are placed, forming an instance of the substate CDD. By Lemma 7.5, the game will therefore result in either the all-cooperator steady state or a limit cycle in which defection is only present in the form of oscillation clusters. The presence of the substate CDD consequently guarantees fixation of the cooperation strategy. □

7.3.3 Fixation probability of the defection strategy

Recall that the nature of the ESS is such that defection has a large probability of persisting. In fact, as long as the length of a run of defectors formed by introducing k defectors into a cycle of

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$n-k$ cooperators is odd, the strategy of defection will persist in the form of an oscillation cluster, by Lemma 7.4. Recall also that remaining oscillation clusters are not considered as fixation as the general trend of the strategy of defection is to decline (despite one round of growth for singleton defectors), and these oscillation clusters in general do not cover the majority of the cycle. The fixation probability of the strategy of defection is therefore the probability of no persistent cooperation in the ESS on a cycle with k defectors and $n-k$ cooperators, for some $k \in \mathbb{N}$, distributed randomly along the cycle in the initial game round.

In order to investigate the fixation probability of the strategy of defection, consider placing k defectors in $n-k$ containers along the cycle, one to the right of each cooperator. As upper bound on the fixation probability, consider the probability of no occurrences of the substate CDD and no occurrences of the substate CCC.

Theorem 7.4 (Fixation probability of the defection strategy).
The fixation probability $F_c^D(n,k)$ of a group of k mutant defectors in the ESS on a cycle of order n with pay-off parameter values satisfying $T < 2S < S+1$ satisfies

$$F_c^D(n,k) \leq \frac{\binom{n-k}{k}}{\binom{n-1}{k}} \tag{7.11}$$

and

$$F_c^D(n,k) \leq \frac{\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{n-3i-1}{k-1}}{\binom{n-1}{n-k}}. \tag{7.12}$$

Proof: The upper bound in (7.11) is the probability of no occurrences of the substate DD, which coincide with occurrences of the substate CDD as the defection run containing DD must at some point be broken by a cooperation run and will therefore contain the substate CDD. This probability is the number of placements $\binom{n-k}{k}$ of k defectors in $n-k$ containers along the cycle, one to the right of each cooperator, *without* replacement divided by the number $\binom{n-1}{k}$ of placements (including occurrences of DD) of k defectors in $n-k$ containers along the cycle *with* replacement.

The upper bound in (7.12) is the probability of no occurrences of the substate CCC. This is the number of ways in which $n-k$ cooperators can be distributed among k containers along the cycle, one to the right of each defector, without ever placing at least three cooperators in the same container divided by the total number of ways of placing these cooperators without restriction. The number of ways of distributing these cooperators with the restriction in place is number of integer solutions to the equation

$$x_1 + x_2 + x_3 + \cdots + x_k = n-k.$$

This number is also the coefficient of z^{n-k} in the generating function

$$(z^0 + z^1 + z^2)^k = (1-z^2)^k\left(\frac{1}{1-z}\right)^k. \tag{7.13}$$

The coefficient is

$$d_{n,k} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{n-3i-1}{k-1}. \tag{7.14}$$

This number is divided by the number $\binom{k+n-k-1}{n-k} = \binom{n-1}{n-k}$ of distributions of $n-k$ cooperators among k containers with replacement to yield the probability of no occurrences of the substate CCC. □

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The upper bounds in (7.11) and (7.12) on the fixation probability of the strategy of defection is plotted in Figure 7.4 for $n \in \{20, \ldots, 50\}$ and $k \in \{4, \ldots, 20\}$. Note that each of the upper bounds dominates in a different region, together forcing the fixation probability to be very small — certainly smaller than 1, which means that the strategy of cooperation is favoured over the strategy of defection in Region C.

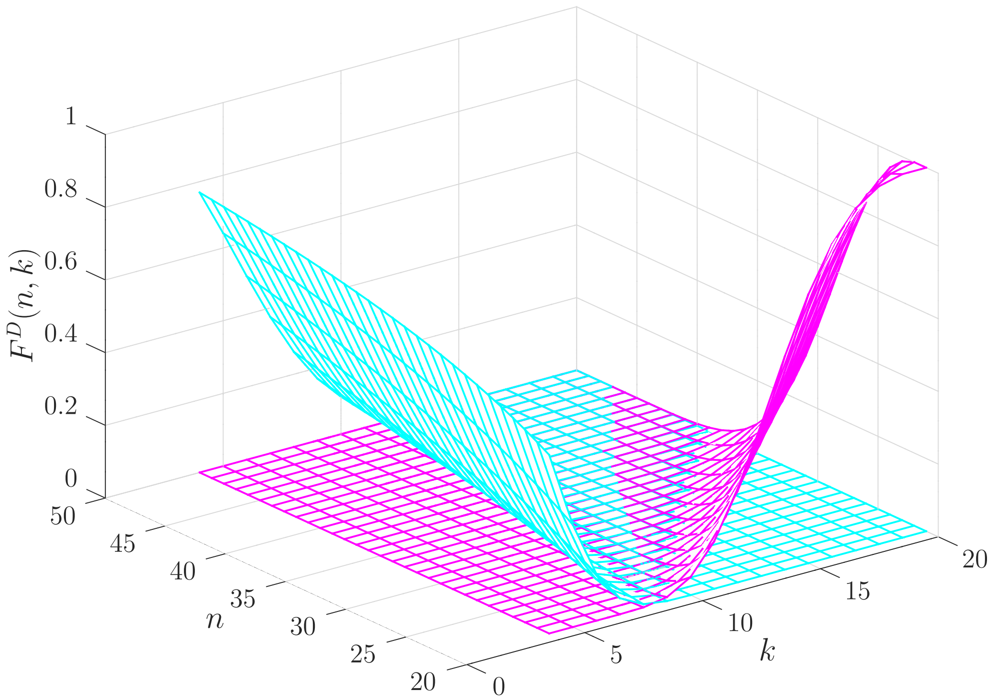

FIGURE 7.4: *Upper bounds on the fixation probability of k defectors in the ESS on a cycle of order n in Region C. The upper bound in (7.11) is plotted in cyan and the upper bound in (7.12) is plotted in purple.*

7.4 State graph component enumeration

The number of components in the state graph of the ESS in Region C is determined by the number of steady states, limit cycles and transient states. As in Region B and the discussion on its state graph in §6.5, Region C poses some challenges to an exact enumeration.

In Region C, it follows from Lemma 7.5 that any initial states containing the substates CDD and/or CCC result in either the all-cooperator steady state or a limit cycle consisting of oscillation clusters dispersed among cooperation runs of sufficient length. This sufficiency of the length of the cooperation runs depends on the length of the next defection cluster in that direction from the singleton defector or oscillation cluster in question. Any defection cluster of length at least 2 will shrink in length during the following game round, thus providing the cooperation cluster in question with an additional cooperator for that game round. Any singleton defector will remove a cooperator from the cooperation cluster in question. Let the phase in which an oscillation cluster has length one, be labelled *phase one* and the phase in which it has length three, *phase two*. Thus, in order to maintain a balance of at least one cooperator in each cooperation cluster,

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it is required that there be at least three cooperators between phase one oscillation clusters and at least one cooperator between phase two oscillation clusters. Furthermore, between phase one and phase two oscillation clusters at least one cooperator is required.

The number of components in the state graph is the sum of the number of steady states and the number of limit cycles that can emanate from initial game states on a cycle of order n. The following two results provide this number of limit cycles for the cases in which each oscillation cluster is either in-phase or exactly alternating in phase.

Each limit cycle which contains i oscillation clusters, all of which are in-phase, has the form

$$\underbrace{DCCC\cdots}_{\text{structure }1}\underbrace{DCCC\cdots}_{\text{structure }2}\cdots\underbrace{DCCC\cdots}_{\text{structure }i}. \tag{7.15}$$

Lemma 7.7 (Enumeration of in-phase steady states).
In the ESS on a cycle of order $n \geq 4$, with pay-off parameters S and T satisfying $T < 2S < S+1$, the number of limit cycles in the state graph in which each oscillation cluster is in-phase (i.e. of the form (7.15)), is given by

$$\begin{aligned} Q^c_{in}(n) = \sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \frac{1}{2i}\Bigg\{ &\binom{n-3i-1}{i-1} + \sum_{j\in\mathcal{R}} \binom{d+\frac{n-4i}{i}d-1}{d-1} + i \pmod 2 \left[i \sum_{m=0}^{\lfloor \frac{n-4i}{2} \rfloor} \binom{\frac{i-1}{2}+m-1}{m} \right] \\ &+ (i+1) \pmod 2 \left[\frac{i}{2} \sum_{m=0}^{\lfloor \frac{n-4i}{2} \rfloor} \binom{\frac{i-2}{2}+m-1}{m} (n-4i-2m+1) \right. \\ &\left. + \frac{i}{2} \binom{\frac{i}{2}+\frac{n-3i}{2}-1}{\frac{i}{2}-1} \right] \Bigg\}, \end{aligned} \tag{7.16}$$

where $d = \gcd(i,j)$ and $\mathcal{R}$ is the set $\{x \in \mathbb{N} \mid i \text{ divides } n\gcd(i,x) \text{ and } x < i\}$.

The proof of the above lemma is similar to that of Lemma 6.8 and is deferred to Appendix A for the sake of brevity in the present discussion.

TABLE 7.2: *$Q^c_{in}(n)$, the number of limit cycles of the ESS on a cycle of order n in which each oscillation cluster is in-phase, for values of $n \in \{4, \dots, 36\}$.*

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$Q^c_{in}(n)$	1	1	1	1	2	2	3	3	5	5	7	8	11	12	17	19	27	31

22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
42	50	69	83	112	139	188	235	319	404	546	703	947	1 232	1 663

Limit cycles may also assume the form

$$\underbrace{DDD\overbrace{C\cdots}^{\text{box }1}D\overbrace{C\cdots}^{\text{box }2}}_{\text{structure }1}\ \underbrace{DDD\overbrace{C\cdots}^{\text{box }3}D\overbrace{C\cdots}^{\text{box }4}}_{\text{structure }2}\ \cdots\ \underbrace{DDD\ \overbrace{C\cdots}^{\text{box }2i-1}\ D\overbrace{C\cdots}^{\text{box }2i}}_{\text{structure }i}. \tag{7.17}$$

This form has the property that each oscillation cluster is alternating in phase. This allows for a denser packing of oscillation clusters along the cycle as the sharing of cooperators allows for the minimum required cooperation run length between two alternating phase oscillation clusters in Region C to be one. These limit cycles are enumerated in the following lemma.

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Lemma 7.8 (Enumeration of alternating phase limit cycles).
For the ESS on a cycle of order $n \geq 4$, with pay-off parameters S and T satisfying $T < 2S < S+1$, the number of limit cycles in the state graph in which the phase of each oscillation cluster is alternating (i.e. of the form (7.17)), is

$$Q^c_{alt}(n) = \sum_{i=1}^{\lfloor \frac{n}{6} \rfloor} \frac{1}{2i} \left[\binom{n-4i-1}{2i-1} + \sum_{j \in \mathcal{L}} \binom{\frac{n-4i}{i}d - 1}{2d-1} + i \binom{\frac{n-4i}{2} - 1}{i-1} (n+1) \pmod 2 \right],$$

where $\mathcal{L}$ is the set $\{x \in \mathbb{N} \mid i$ divides $n \gcd(i,x)$ and $x < i\}$ and $d = \gcd(i,j)$.

The proof of Lemma 7.8 is similar to that of Lemma 6.9 and is again deferred to Appendix A. Lemmas 7.7 and 7.8 enumerate the limit cycles in Region C in which the oscillation clusters are either all in-phase or all alternating in phase. As no limit cycle is thus counted twice, their sum forms a lower bound on the total number of limit cycles and steady states possible, which is identical to the number of components in the state graph of Region C.

Table 7.3: $Q^c_{alt}(n)$, *the number of limit cycles of the ESS on a cycle of order n in which each oscillation cluster is alternating in phase, for values of $n \in \{6, \ldots, 36\}$.*

n	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$Q^c_{alt}(n)$	1	1	2	2	3	3	5	5	9	10	17	20	32	38	58	73	109	142

24	25	26	27	28	29	30	31	32	33	34	35	36
213	283	416	565	820	1 124	1 628	2 256	3 256	4 558	6 558	9 233	13 266

The following theorem describes the relationship more clearly.

Theorem 7.5 (Lower bound on the number of components in the state graph).
For the ESS on a cycle of order $n \geq 4$, with pay-off parameters S and T satisfying $T < 2S < S+1$, the number $Q^c(n)$ of components in the state graph satisfies

$$Q^c(n) \geq Q^c_{in}(n) + Q^c_{alt}(n) + 2. \tag{7.18}$$

Proof: There is no overlap in the limit cycles of in-phase and alternating phase oscillation clusters and so their numbers may be summed. The addition of the constant 2 in (7.18) is accounted for by the all-cooperator and all-defector steady states, each in a component of their own. Lastly, there may be limit cycles in which the phases of the oscillation cluster are neither all in-phase nor all alternating, and these limit cycles have not been counted. Therefore the addition of $Q^c_{in}(n)$, $Q^c_{alt}(n)$, and 2 serves as a lower bound on the total number of components in the state graph of Region C. □

The lower bounds in Theorem 7.5 are tabulated in Table 7.4 and are also plotted in Figures 7.5 and 7.6 on linear and log-linear axes, respectively. It is worthwhile noting that as n increases, the impact of the in-phase components on the total number of components diminishes.

7.5 Chapter summary

This chapter was devoted to an in-depth investigation into the ESS temporal dynamics in Region C of the (S,T)-phase plane where the inequality chain $T < 2S < S+1$ holds. The chapter opened in §7.1 with a description of initial states that lead to persistent cooperation in the steady states and limit cycles of the game. More specifically, it was found that the states CCC and CDD each

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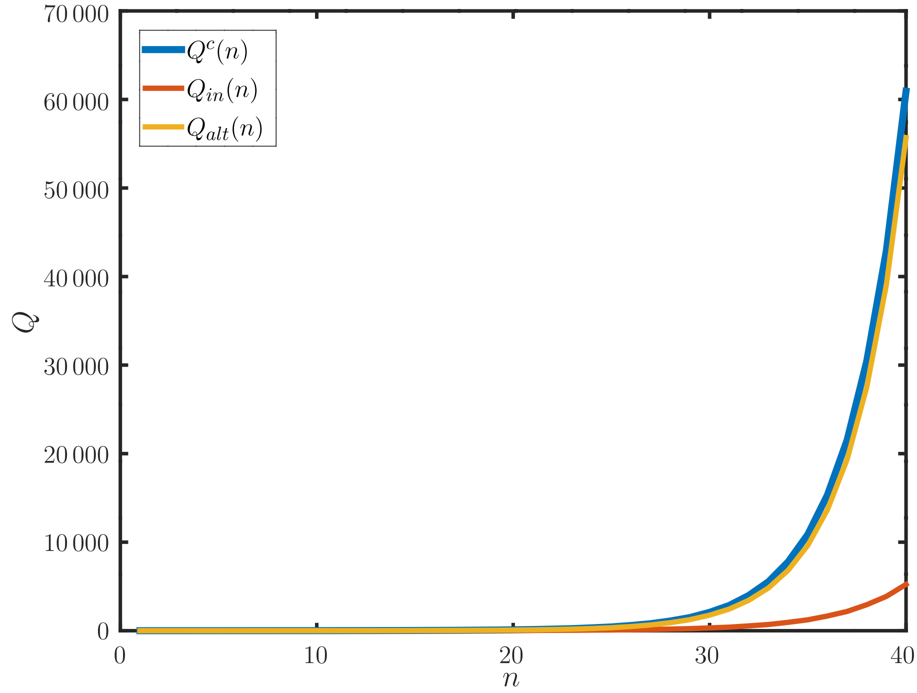

FIGURE 7.5: *The lower bound $Q^c(n)$ of Theorem 7.5 on the size of the state graph of the ESS on a cycle of order n in Region C of the (S, T)-phase plane, on linear axes.*

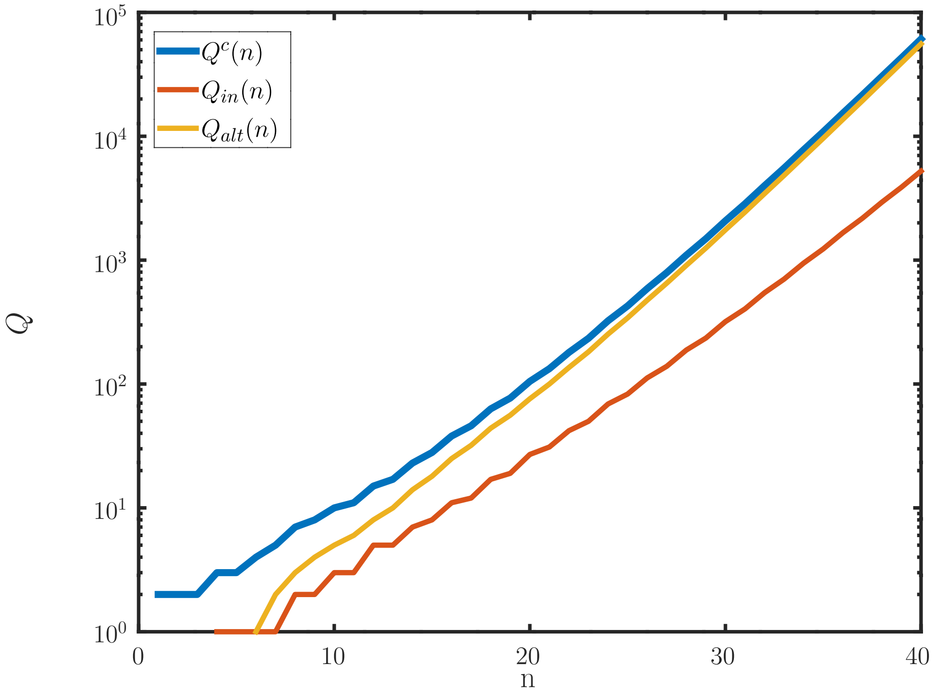

FIGURE 7.6: *The lower bound $Q^c(n)$ of Theorem 7.5 on the size of the state graph of the ESS on a cycle of order n in Region C of the (S, T)-phase plane, on log-linear axes.*

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TABLE 7.4: $Q^c(n)$, *the lower bound on the number of components in the state graph in Theorem 7.5 of the ESS on a cycle of order* $n \in \{1, \ldots, 32\}$ *in Region C of the* (S, T)*-phase plane.*

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$Q^c(n)$	2	2	2	3	3	4	4	6	6	8	8	12	12	18	20	30	34	51

19	20	21	22	23	24	25	26	27	28	29	30	31	32
59	87	106	153	194	284	368	530	706	1 010	1 361	1 949	2 662	3 804

guarantees persistent cooperation. These sufficient conditions for persistent cooperation were subsequently utilised in §7.2 to determine the probability that a randomly generated initial state of the ESS on a cycle of order n leads to persistent cooperation. The limit of this probability as $n \to \infty$ was also evaluated. A deterministic incarnation of the notion of the fixation probabilities of cooperation and defection in the context of Region C was investigated in §7.3. The outcome of the analysis was that the strategy of cooperation is favoured over that of defection in Region C of the ESS on a cycle of order n. Finally, lower bounds were established in §7.4 on the number of components in the state graph.

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CHAPTER 8

Conclusion

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This chapter opens in §8.1 with a brief summary of the contents of this thesis, with an emphasis on the results obtained during an analysis of the ESS in the various regions of the (S, T)-phase plane. The contributions of the thesis are highlighted thereafter in §8.2, while §8.3 is devoted to the suggestion of a number of avenues of potential future follow-up work on the research documented in this thesis.

8.1 Thesis summary

The contents of this thesis is summarised in this section in a chapter by chapter fashion, starting with Chapter 2 and ending with Chapter 7.

A number of mathematical prerequisites used throughout the thesis were reviewed in Chapter 2 with a view to self-containment of the work documented in the thesis as far as this is possible. Basic graph theoretic concepts and nomenclature were presented in order to elucidate the method of representing the structure among players of the ESS as a cycle graph. A number of notions related to digraphs and the transfer matrix method were also described. The combinatorial concept of a generating function was reviewed briefly so as to allow for an exposition of the transfer matrix method as a tool for counting strings not containing certain specified substrings. Lastly, the underlying requirements for a basic understanding of the Cauchy-Frobenius Lemma were discussed, including the notions of groups and group actions. The chapter closed with a discussion on the use of the Cauchy-Frobenius Lemma and an example application thereof for the purpose of counting the number of bracelets of six beads in two colours.

An overview of the literature pertaining to the work in this thesis was presented in Chapter 3. This literature review was an attempt at fulfilment of Objective I in §1.4. The chapter opened

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with a brief introduction to the field of classical game theory with special attention afforded to two important 2×2 games — the prisoner's dilemma and the snowdrift game. Quintessential work on iterated games was also reviewed, paving the way for the development of an understanding of basic concepts in evolutionary game theory. Important work in the fields of spatial and graphical games was highlighted, differentiated solely by the type of graphs upon which they are played. Honing in on the area of work pertaining to this thesis, the chapter closed with a brief review of the literature dedicated specifically to evolutionary games on cycles.

The mathematical representation of the ESS adopted in this thesis was described in Chapter 4, highlighting the use of graphs to represent player populations in a game theoretic setting. The content of this chapter was aimed at revising a previously established mathematical framework for analysing the deterministic ESS, thereby fulfilling Objective II of §1.4. The pay-off parameters of the ESS were normalised to allow for a more managable investigation into the game dynamics involving the variation of only two pay-off parameters instead of four. The result of this normalisation was the (S, T)-phase plane of the ESS, depicted in Figure 4.5 as containing three regions representing distinct, interesting long-term asymptotic game dynamics. The notion of a fixation probability was also discussed and a suitable variation thereof identified for inclusion in investigations witin the deterministic setting of evolutionary games, in partial fulfilment of Objective IV in §1.4.

Chapters 5 to 7 contain results obtained during a long-term asymptotic analysis of the ESS dynamics in various regions of interest of the (S, T)-phase plane, identified in Chapter 4. The work documented in these chapters stands in fulfilment of Objectives III–IV of §1.4.

In Chapter 5, it was shown that the asymptotic behaviour of the ESS in Region A is identical to that of the ESPD in a particular region of interest considered by Burger *et al.* [3] in 2012. This meant that the requirements for, and the probability of, persistent cooperation were simply those pertaining to the ESPD and, as such, these were merely restated in the remainder of the chapter. Fixation probabilities are not applicable to Region A as there is no opportunity for spatial growth of the strategy of cooperation and so there is no probability of fixation of the strategy of cooperation in this region. The strategy of defection is, however, associated with a non-zero fixation probability and is thus favoured over the strategy of cooperation in Region A of the ESS on a cycle. Finally, the number of components in the state graph of the ESS on a cycle for Region A was restated from Burger *et al.* [3].

Chapter 6 was devoted to an investigation of the ESS in a region of the (S, T)-phase plane in which the first opportunity for cooperation growth presents itself, Region B. It was shown that a necessary and sufficient requirement for persistent cooperation in this region is the presence of the partial game states $CCDD$ and/or $CCCC$. This characterisation was used in conjunction with the matrix transfer method to compute the probability of persistent cooperation in the ESS on a cycle from a randomly generated initial game state in Region B of the phase plane. It was also shown that this probability approaches unity as the order of the cycle increases. The fixation probabilities of the strategies of cooperation and defection were investigated and it was shown that the strategy of cooperation is favoured over that of defection in Region B of the ESS on a cycle. Using the Cauchy-Frobenius Lemma, lower bounds on the number of components in the state graph of the ESS were established by enumerating the components in which all oscillation clusters are either in-phase or alternating in phase.

The ESS dynamics in Region C of the (S, T)-phase plane, in which the pay-off parameters satisfy $T < 2S < S + 1$, was the topic of investigation in Chapter 7. The nature of the game in this region induces a new opportunity for the strategy of cooperation to win over that of defection because of the inequality $2S > T$. A similar analysis to the one carried out in the previous chapter for Region B was conducted with the key difference being an abundant dominance of

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the cooperation strategy in Region C. A necessary and sufficient requirement for persistent cooperation was found to be the presence of the partial states CDD and/or CCC within the initial game state. This characterisation was again used to compute the probability of persistent cooperation for Region C of the ESS on a cycle. The synchronous and deterministic version of fixation probability for the strategy of cooperation is determined by the probability of either of the aforementioned partial states. Interestingly, this fixation probability of the strategy of cooperation is unity if the cardinality k of the mutant strategy set is at most $n/2$, where n is the order of the underlying cycle. The fixation probability of the strategy of defection was defined as the probability of no persistent cooperation for a fixed value of k and upper bounds were established on this probability, showing that it remains strictly below 1 for $k < n/2$. The conclusion was that the strategy of cooperation is favoured in the ESS on a cycle in Region C over that of defection. The final element in the investigation of the ESS in Region C was the establishment of a lower bound on the number of components in the state graph. This lower bound was found by enumerating limit cycles consisting of a combination of in-phase oscillation and clusters oscillation clusters that are alternating in phase.

8.2 Appraisal of thesis contributions

The contributions of this thesis are described in this section, highlighting the results obtained during the study. The investigation documented in this thesis is the first investigation into the ESS on a cycle (specifically with deterministic and global updating) and, as such, the contributions listed below are all novel to the best of the author's knowledge.

Contribution 1 *Necessary and sufficient requirements for, and the probability of, persistent cooperation in the ESS on a cycle.*

An analytical characterisation of initial game states that lead to persistent cooperation was presented and this characterisation, in conjunction with the matrix transfer method, was used to compute the probability of persistent cooperation from a randomly generated initial game state for all three regions of the ESS phase plane.

Contribution 2 *Formulation and investigation of a deterministic variation on fixation probability.*

The concept of fixation probability is well established and widely used in the analysis of evolutionary games. The context in which this concept is applied has always been stochastic update rules which allow for the fixation of non-winning strategies. A new definition of fixation probability was proffered in this thesis which pertains to the analysis of deterministic games. Furthermore, this variation on the notion of a fixation probability was investigated within two of the regions of the ESS phase plane in which growth of cooperation is possible, showing that the strategy of cooperation is favoured over that of defection in the ESS on a cycle.

Contribution 3 *Bounds on the number of components in the state graph of the ESS on a cycle.*

Bounds on the numbers of components in the state graph of the ESS on a cycle in Regions B and C of the (S, T)-phase plane have been established using the Cauchy-Frobenius Lemma. The components thus enumerated account for limit cycles in which each of the oscillation clusters are in-phase with one another or are else alternating in phase.

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8.3 Suggestions for future work

The avenues for possible future work following on the research documented in this thesis are abundant and varied. A complete enumeration of the ESS state graph components for Regions B and C may be tractable for an experienced and dedicated mathematician versed in the art of enumerative combinatorics. Investigations into the dynamics of other games in the similar setting of a cyclic player structure may also be of interest as most of the work in this regard has been in the context of stochastic update rules. Pursuing the dynamics of these games on more intricate population structures would add a dimension of reality that is indeed missing in the cyclic population structure.

8.3.1 Better lower bounds on the state graph enumeration

In Regions B and C of T-S phase plane of the ESS on a cycle, only lower bounds were established on the number of components of the state graph. An improvement of this lower bound in Region B, for example, may be made by counting the number of states of the form

$$\overbrace{\underbrace{XXX}_{\text{o.c. }1}\underbrace{CC\ldots}_{\text{run }1}}^{\text{box }1}\overbrace{\underbrace{XXX}_{\text{o.c. }2}\underbrace{CC\cdots}_{\text{run }2}}^{\text{box }2}\cdots\overbrace{\underbrace{XXX}_{\text{o.c. }i}\underbrace{CC\ldots}_{\text{run }i}}^{\text{box }i}, \tag{8.1}$$

in which the substate XXX represents an oscillation cluster alternating between the substates CDC and DDD. The form (8.1) cannot contain as many oscillation clusters as (6.22) due to the number of cooperators required between oscillation clusters, but it does allow for more freedom in the phases of oscillation clusters.

Lemma 8.1 (Alternative bounds on the state graph in Region B).
The number of transient states and states that form part of limit cycles of the form (8.1) in the state graph of the ESS on a cycle of order n with pay-off parameter values satisfying $2S < T < S+1$ is

$$\begin{aligned} Q^b_{opt}(n) = \sum_{i=1}^{\lfloor \frac{n}{5} \rfloor} \frac{1}{2i}\Bigg\{ & 2^i\binom{n-4i-1}{i-1} + \sum_{j\in\mathcal{T}} 2^d \binom{\frac{(n-5i)d}{i}+d-1}{d-1} \\ & + \frac{i}{2} 2^{\frac{i+2}{2}} \binom{\frac{n-4i}{2}-1}{\frac{i}{2}-1} (i+1) \pmod 2 \\ & + \frac{i}{2} \sum_{p=0}^{\lfloor \frac{n-5i}{2} \rfloor} \left[2^{\frac{i}{2}} (n-5i-2p+1) \binom{\frac{i-2}{2}+p-1}{p} (i+1) \pmod 2 \right] \\ & + i \sum_{q=0}^{\lfloor \frac{n-5i}{2} \rfloor} 2^{\frac{i+1}{2}} \binom{\frac{i-1}{2}+q-1}{q} (i) \pmod 2 \Bigg\}, \end{aligned} \tag{8.2}$$

where $\mathcal{T} = \{x \in \mathbb{N} \mid i$ divides $n \gcd(i,x)$ and $x < i\}$ and $d = \gcd(i,j)$.

The proof of Lemma 8.1 is given in Appendix B. Here $Q^b_{opt}(n)$ represents the number of states that form part of a pair of states constituting a limit cycle, and others that are in a "limit cycle" with themselves forming transient states (limit cycles of length two in which both states are isomorphic to one another). Denote the number of states that form part of limit cycles by y and the number of transient states by x. Then $Q^b_{opt}(n) = y + x$, and the number of components countable in this way is $y/2 + x$. Therefore, $Q^b_{opt}(n)/2 \le y/2 + x$ is a lower bound

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on the number of components in the state graph. Future work in this regard may consist of enumerating x precisely and finding an even larger lower bound $(Q^b_{opt}(n) - x)/2 + x$ on the number of components in the state graph.

The enumeration of x would involve counting states of the form (8.1) with the property that when the phases of the oscillation clusters shift, the resulting states are automorphic to the original one. A similar improvement may be applied to the lower bound on the number of components in the state graph for Region C of the (S, T)-phase plane.

8.3.2 Alternative population structures

The population structure in the evolutionary game studied in this thesis has been limited to cycles. This is indeed a convenient starting point — a simple structure that lends itself to analytical investigation yet exhibits complicated dynamics in the update rules (synchronicity and deterministic updating). There are various natural extensions to this simple population structure.

An investigation into the ESS on the circulant $C_n\langle 1, 2\rangle$ could yield interesting results as this would add the formation of clusters[1] which are completely absent in cycle graphs (with the exception of $n = 3$). Figure 8.1 contains a graphical representation of the circulant $C_8\langle 1, 2\rangle$. An investigation similar to the one carried out in this thesis was conducted in the context of the ESPD played on circulants of the form $C_n\langle 1, 2\rangle$ and toroidal grids by Landman [13]. As such, an investigation into the ESS dynamics on a circulant would be valuable for the sake of completeness.

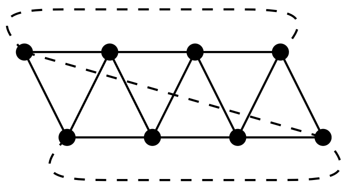

FIGURE 8.1: *The circulant $C_8\langle 1, 2\rangle$, a natural extension to the cycle graph C_8 obtained by extending each player's neighbourhood by one vertex on either side.*

A second natural extension to the cycle graph is the ladder graph, depicted in Figure 8.2. Each player receives a third neighbour, although there is still no clustering in this graph and so the results might be comparable to those for the cycle as underlying graph.

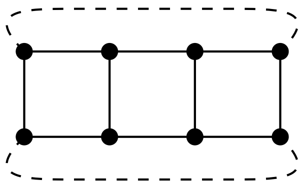

FIGURE 8.2: *The ladder graph which duplicates the cycle and joins players in the same positions of the respective cycles to one another.*

Finally, toroidal grids also present an extension to the cyclic incarnation of the game considered in this thesis as they add a dimension which also features wrapping like that present in a cycle. The investigation into the ESPD conducted by van der Merwe [4] culminated in a study of the

[1] Triplets of vertices all adjacent to one another.

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ESPD on toroidal grids. This work was then taken further by Landman [13]. An example of a toroidal grid is illustrated graphically in Figure 8.3.

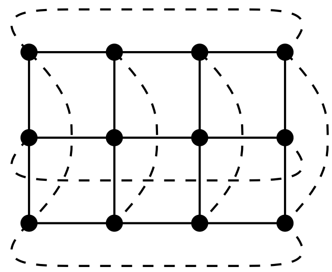

FIGURE 8.3: *A toroidal grid graph of order 12, which is a grid featuring wrapping around the sides (top and bottom).*

An investigation into the ESS on toroidal grids may yield interesting results with an added complexity to the population structure. In small instances of toroidal grids, there is some clustering which disappears as the grids become larger. This dynamic is certainly of interest as it has been claimed that clustering plays a role in hindering the growth of cooperation in the ESS [29] in the case of adopting stochastic update rules, while with deterministic imitation rules the growth of cooperation is robust. Clarifying this dynamic further by investigating the differences between graphs with clustering and similar graphs without clustering may be of interest.

8.3.3 The remaining 2×2 games

The ESPD on a cycle has been investigated by Burger *et al.* [4] and inspired the work documented in this thesis on the ESS. There are a variety of 2×2 games which should lend themselves to a similar investigation. In the (S,T)-phase plane, a variety of games may be found by relaxing the inequality chain $P < S < R < T$, including the prisoner's dilemma (which has been studied extensively) and the stag-hunt game. Other games are not often studied as they may be perceived to be lesser social dilemmas and are, therefore, considered less interesting.

The stag-hunt game results from the inequality chain $S < P \leq T < R$. The game may be described as a hunt in which each player may opt to hunt alone and obtain (possibly share) a hare, or hunt together and share a stag (which cannot be hunted alone). The social dilemma arises due to a trade-off between security and cooperation. Hauert [9], for example, studied the stag-hunt game in a different setting than that considered in this thesis. The study of a cyclic population structure and its effect on the persistence of cooperation in the evolutionary spatial stag-hunt game would not be amiss in the setting of deterministic and global updating.

8.3.4 Temporal game dynamics on the isoclines of the (S,T)-phase plane

The investigation documented in this thesis focused solely on the regions of inequality in the (S,T)-phase plane. The game dynamics on the lines of equality in the (S,T)-phase plane, or isoclines, is another possible extension of the work conducted. The game dynamics on the line $T = S + 1$ will be similar to those in Region A, with different numbers of cooperators and defectors required in adjacent runs to ensure persistent cooperation. There are no oscillation clusters in the state graph on this isocline as in Regions B and C, because a run of three defectors adjacent to two cooperators will result in a standoff due to the equality $T = S + 1$. The isocline $T = 2S$ should, however, indeed admit oscillation clusters as runs of three defectors adjacent to

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at least two cooperators on each side will reduce to a singleton defector due to the inequality $T < S+1$ and this singular defector will subsequently obtain the largest pay-off achievable in the game, inducing an oscillation cluster. Furthermore, there should be some steady states including singleton cooperators dispersed amongst runs of at least two defectors. A full investigation into the game dynamics of the ESS on the isoclines should therefore yield interesting results and provide a fuller picture of the ESS on a cycle in general.

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APPENDIX A

Proofs of Lemmas 7.7 and 7.8

Two enumeration results for the ESS in Region C are restated in this appendix and also proved. Let $\mathcal{X}_i$ denote the number of limit cycles of the form

$$\underbrace{DDDC\cdots}_{\text{structure }1}\underbrace{DDDC\cdots}_{\text{structure }2}\cdots\underbrace{DDDC\cdots}_{\text{structure }i} \tag{A.1}$$

Lemma 7.7 (Enumeration of in-phase steady states).
In the ESS on a cycle of order $n \geq 4$, with pay-off parameters S and T satisfying $T < 2S < S+1$, the number of limit cycles in the state graph in which each oscillation cluster is in-phase (i.e. of the form (7.15)), is given by

$$\begin{aligned}\mathcal{Q}^c_{in}(n) = \sum_{i=1}^{\lfloor\frac{n}{4}\rfloor}\frac{1}{2i}\Bigg\{\binom{n-3i-1}{i-1}+\sum_{j\in\mathcal{R}}\binom{d+\frac{n-4i}{i}d-1}{d-1}+i \pmod 2\Bigg[i\sum_{m=0}^{\lfloor\frac{n-4i}{2}\rfloor}\binom{\frac{i-1}{2}+m-1}{m}\Bigg]\\ +(i+1) \pmod 2\Bigg[\frac{i}{2}\sum_{m=0}^{\lfloor\frac{n-4i}{2}\rfloor}\binom{\frac{i-2}{2}+m-1}{m}(n-4i-2m+1)\\ +\frac{i}{2}\binom{\frac{i}{2}+\frac{n-3i}{2}-1}{\frac{i}{2}-1}\Bigg]\Bigg\},\end{aligned}$$

where $d = \gcd(i,j)$ and $\mathcal{R}$ is the set $\{x \in \mathbb{N} \mid i$ divides $n \gcd(i,x)$ and $x < i\}$.

Proof: Clearly $\sum_{i=1}^{\lfloor\frac{n}{4}\rfloor}\mathcal{X}_i$ is the total number of components in the state graph containing in-phase oscillation clusters and therefore provides a lower bound on the number of components in the state graph. There are $n-5i$ cooperators that can be distributed among the i structures in (A.1) and this leads to an enumeration problem which can be solved by applying the Cauchy-Frobenius Lemma.

Let $\mathcal{I}$ be the set of limit cycles of the form described above and let ι be the identity operator in the group $\mathcal{U}$ which acts on these states. Furthermore, let ρ^j be the permutation in which each structure is shifted j positions to the right in a modular fashion, and let δ be the action which reflects the structures about a diametrical axis passing through the first structure. The effect of the action δ^j depends on the parity of i. For odd i, the action δ^j reflects the structures about the diametrical axis passing through structure j. For even i, the action δ^j reflects the structures around the diametrical axis passing through structures j and $(j+i/2) \pmod i$ for $j = 1,\ldots,i/2$, while for $j = i/2,\ldots,i$ the action δ^j reflects the structures about the diametrical

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axis passing between structures $j+1 \pmod{i/2}$ and $j+2 \pmod{i/2}$. The group $\mathcal{U}$ is formed by the set $\{\iota, \rho^1, \rho^2, \ldots, \rho^{i-1}, \delta, \delta^2, \ldots, \delta^i\}$, which has order $2i$. According to the Cauchy-Frobenius Lemma,

$$\mathcal{X}_i = \frac{1}{|\mathcal{U}|} \sum_{u \in \mathcal{U}} |F_u|, \tag{A.2}$$

where F_u denotes the set of states in $\mathcal{I}$ that are fixed by u.

For the identity operator, which maps each structure to itself and therefore fixes each element of $\mathcal{I}$, this number is

$$|F_\iota| = \binom{i+n-4i-1}{n-4i} = \binom{n-3i-1}{i-1}, \tag{A.3}$$

which is the number of ways of distributing $n-4i$ cooperators among the i structures in (A.1) with replacement.

Shifting each structure j positions to the right by means of the action ρ^j fixes those states in which the first j structures determine the remaining $i-j$ structures, as long as $j|i$. Alternatively, the first $\gcd(i,j)$ structures determine the remaining structures if j does not divide i (which incidentally is j if $i|j$). Setting $d = \gcd(i,j)$ means that the number of states fixed under F_{ρ^j} is given by

$$\binom{(n-4i)d/i + d - 1}{d-1}.$$

This is the number of ways in which $(n-4i)d/i$ cooperators can be distributed among d structures, which can only occur if the number of cooperators is sufficient to continue the pattern i/d times, and this is precisely the case when $i|(n \gcd(i,j))$.

The effect of the action δ is that the structures are swapped with one another in pairs, leaving either only the first structure in place if i is odd, or leaving both the first and middle (1^{st} and $\frac{i}{2}^{\text{th}}$) structures in place if i is even. This effect can be seen in Figure 6.6. Thus the number of states fixed by the action δ is the number of possible states in which each pair of switched structures is identical. For even i, distributing m cooperators amongst the first half of the structures that switch ($\frac{i-2}{2}$) determines the distribution of $2m$ cooperators in all of the structures, except for the two that are fixed to themselves. The remaining $n-4i-2m$ cooperators can then be distributed amongst the two remaining structures in

$$\binom{n-4i-2m+1}{1}$$

ways. In total that is

$$\binom{\frac{i-2}{2}+m-1}{m}(n-4i-2m+1)$$

arrangements for even i. For odd i, the distribution of m cooperators among the $\frac{i-1}{2}$ structures determines the distribution of $2m$ cooperators amongst $i-1$ structures. The remaining cooperators must then be distributed among the first structure, which can only be done in one way. This totals

$$\binom{\frac{i-1}{2}+m-1}{m}$$

arrangements for odd i.

Adding these two values together and ensuring that the correct instances are added by multiplication of $i-1 \pmod 2$ and $i \pmod 2$, respectively, yields the total number of arrangements,

$$|F_\delta| = \sum_{m=0}^{\lfloor \frac{n-4i}{2} \rfloor} (n-4i-2m+1)\binom{\frac{i-2}{2}+m-1}{m}(i-1) \pmod 2 + \binom{\frac{i-1}{2}+m-1}{m} i \pmod 2. \tag{A.4}$$

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For odd i and for even i (but only for $j = 1, \ldots, i/2$), the effect of the action δ^j is that the structures are reflected about the diametrical axis passing through the j^{th} structure. For odd i, this always fixes one structure in place and switches the remaining $\frac{i-1}{2}$ in pairs, while for even i, two of the structures map to themselves and the remaining $\frac{i-2}{2}$ structures form pairs that switch. The number of ways in which this can be done for each j in question is the same as for the action δ.

For even i and for $j = i/2, \ldots, i$, the effect of the action δ^j is a reflection about the diametrical axis passing between two structures, starting between structures one and two, and ending between structures $i/2 - 1$ and $i/2$. This means that all the structures form pairs which map to one another. Thus, for even i, the number of arrangements fixed by δ^j (for $j = 1, \ldots, i/2$) is given by

$$\binom{\frac{i}{2} + \frac{n-4i}{2} - 1}{\frac{n-4i}{2}} = \binom{\frac{n-3i}{2} - 1}{\frac{i}{2} - 1},$$

which is the number of possible distributions of $\frac{n-4i}{2}$ cooperators among $\frac{i}{2}$ structures with replacement.

Therefore the number of steady states fixed by $F_{\delta^1}, \ldots, F_{\delta^i}$ is given by

$$\begin{aligned}\sum_{j=1}^{i} |F_{\delta^j}| =& \frac{i}{2}\binom{\frac{n-3i}{2} - 1}{\frac{i}{2} - 1}(i+1) \pmod 2\\ &+ \sum_{m=0}^{\lfloor \frac{n-4i}{2} \rfloor} \left[\frac{i}{2}(n - 4i - 2m + 1)\binom{\frac{i-2}{2} + m - 1}{m}(i+1) \pmod 2\right.\\ &\left. + i\binom{\frac{i-1}{2} + m - 1}{m} i \pmod 2\right].\end{aligned}$$

□

The limit cycles in which each oscillation cluster is alternating in phase in Region C are of the form

$$\overbrace{\underbrace{DDD}_{\text{run }1}\underbrace{C\ldots}_{\text{run }2}}^{\text{box }1}\overbrace{\underbrace{D}_{\text{run }3}\underbrace{C\cdots}_{\text{run }4}}^{\text{box }2}\cdots\overbrace{\underbrace{DDD}_{\text{run }4i-4}\underbrace{C\cdots}_{\text{run }4i-3}}^{\text{box }2i-1}\overbrace{\underbrace{D}_{\text{run }4i-1}\underbrace{C\cdots}_{\text{run }4i}}^{\text{box }2i}. \tag{A.5}$$

Lemma 7.8 (Enumeration of alternating phase limit cycles).
For the ESS on a cycle of order $n \geq 4$, with pay-off parameters S and T satisfying $T < 2S < S+1$, the number of limit cycles in the state graph in which the phase of each oscillation cluster is alternating (i.e. of the form (7.17)), is

$$Q^c_{alt}(n) = \sum_{i=1}^{\lfloor \frac{n}{6} \rfloor} \frac{1}{2i}\left[\binom{n-4i-1}{2i-1} + \sum_{j\in\mathcal{L}}\binom{\frac{n-4i}{i}d - 1}{2d-1} + i\binom{\frac{n-4i}{2} - 1}{i-1}(n+1) \pmod 2\right],$$

where $\mathcal{L}$ is the set $\{x \in \mathbb{N} \mid i$ divides $n \gcd(i,x)$ and $x < i\}$ and $d = \gcd(i,j)$.

Proof: It is evident from (A.5) that the even boxes map to even boxes for fixation under an action g, and that the odd boxes, similarly, map to odd boxes for the arrangement to be fixed under that action. In (A.5), the number of determined symbols is $6i$, and the remaining $n - 6i$ symbols have to be distributed among $2i$ distinguishable boxes ($2i$ distinguishable cooperation runs). Define $\mathcal{P}$ as the total number of limit cycles of the form (A.5). The action ι is the identity which maps each run to itself. The action ρ^j shifts each run $4j$ positions to the right in a modular fashion. Lastly, the action σ_k reflects the runs around the diametrical axis passing through a

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defection runs of box k and box $k+i$ (runs $2k-1$ and $2(k+i)-1$). The group of actions $\mathcal{H}$ consists of the set $\{\iota, \rho^1, \rho^2, \ldots, \rho^{i-1}, \sigma_1, \sigma_2, \ldots, \sigma_i\}$. According to the Cauchy-Frobenius Lemma,

$$\mathcal{P} = \frac{1}{|\mathcal{H}|} \sum_{h \in \mathcal{H}} |F_h|, \tag{A.6}$$

where F_h is the set of arrangements of the $n-6i$ symbols that remain fixed under the action h. Every arrangement is fixed under the action ι as this action maps each box to itself, and so

$$|F_\iota| = \binom{2i+n-6i-1}{n-6i} = \binom{n-4i-1}{2i-1}. \tag{A.7}$$

The action ρ^j fixes those arrangements in which the first $2\gcd(i,j)$ boxes are free and are then repeated $2i/2\gcd(i,j)$ times. Setting $d = \gcd(i,j)$ means that distributing $\frac{n-6i}{2i}2d$ symbols among $2d$ boxes determines the distribution of the remaining symbols, as long as i divides $n\gcd(i,x)$. Therefore,

$$|F_{\rho^j}| = \binom{2d + \frac{n-6i}{i}d - 1}{2d-1} = \binom{\frac{n-4i}{i}d - 1}{2d-1}. \tag{A.8}$$

Lastly, the action σ_k maps exactly two defection runs to themselves and the remaining $4i-2$ runs are switched in pairs, consistently mapping defection runs of the same length to one another. Assigning $(n-6i)/2$ cooperators to the i cooperation runs from run $2k$ to run $2(k+i)$ determines the distribution of the $n-6i$ cooperators in total among all $2i$ cooperation runs. Therefore, the number of arrangements fixed by σ_k is given by

$$\binom{i + \frac{n-6i}{2} - 1}{\frac{n-6i}{2}} = \binom{\frac{n-4i}{2} - 1}{i-1}. \tag{A.9}$$

This can only be achieved if $n-6i$ is even, which is implied when n is even, and so multiplying by $n+1 \pmod 2$ ensures that this action is only counted when n is even. This is the same for all chosen k which run from 1 to i. Since there are i of these actions,

$$\sum_{k=1}^{i} |F_{\sigma_k}| = i\binom{\frac{n-6i}{2} - 1}{i-1}(n+1) \pmod 2. \qquad \square$$

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APPENDIX B

On the number of components in the state graph

This appendix contains an alternative lower bound on the number of components in the state graph in Region B by allowing for freedom of phases of oscillation clusters in limit cycle states. In principle, the same improvement may be applied to the lower bound of the number of components in the state graph in Region C.

Consider counting the limit cycles of the form (8.1) in Region B. The form (8.1) cannot contain as many oscillation clusters as (6.22) due to the number of cooperators required between oscillation clusters, but it does allow for more freedom in the phase of oscillation clusters.

Lemma 8.1 (Alternative bounds on the state graph in Region B).
The number of transient states and states that form part of limit cycles of the form (8.1) in the state graph of the ESS on a cycle of order n with pay-off parameter values satisfying $2S < T < S+1$ is

$$\begin{aligned} Q^b_{opt}(n) = \sum_{i=1}^{\lfloor \frac{n}{5} \rfloor} \frac{1}{2i} \Bigg\{ & 2^i \binom{n-4i-1}{i-1} + \sum_{j \in \mathcal{T}} 2^d \binom{\frac{(n-5i)d}{i} + d - 1}{d-1} \\ & + \frac{i}{2} 2^{\frac{i+2}{2}} \binom{\frac{n-4i}{2} - 1}{\frac{i}{2} - 1} (i+1) \pmod 2 \\ & + \frac{i}{2} \sum_{p=0}^{\lfloor \frac{n-5i}{2} \rfloor} \left[2^{\frac{i}{2}} (n-5i-2p+1) \binom{\frac{i-2}{2} + p - 1}{p} (i+1) \pmod 2 \right] \\ & + i \sum_{q=0}^{\lfloor \frac{n-5i}{2} \rfloor} 2^{\frac{i+1}{2}} \binom{\frac{i-1}{2} + q - 1}{q} (i) \pmod 2 \Bigg\} \end{aligned} \tag{B.1}$$

where $\mathcal{T} = \{x \in \mathbb{N} \mid i$ divides $n \gcd(i,x)$ and $x < i\}$ and $d = \gcd(i,j)$.

Proof: In (8.1), the number of determined symbols is $5i$, the remaining $n - 5i$ symbols have to be distributed among i distinguishable runs, and the phases of i oscillation clusters have to be determined. Define $\mathcal{U}$ as the total number of steady states of the form (8.1). The action ι is the identity which maps each run to itself. The action ρ^j shifts each box (oscillation cluster and corresponding cooperation run) j positions to the right in a modular fashion. The effect of the action δ^j depends on the parity of i. For odd i, the action δ^j reflects the oscillation clusters

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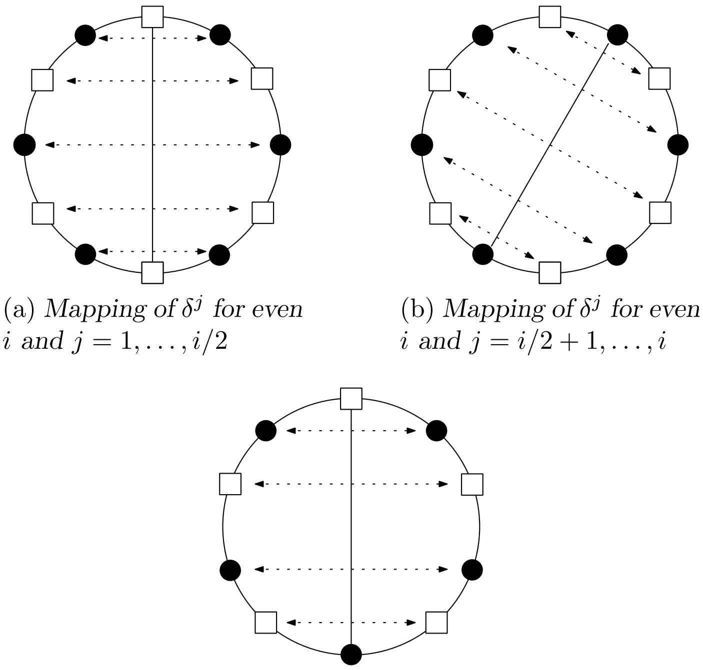
and runs about the diametrical axis passing through oscillation cluster j. For even i and for $j = 1, \ldots, i/2$, the action δ^j reflects the oscillation clusters and runs about the diametrical axis passing through oscillation cluster j, while for $j = i/2, \ldots, i$, the action δ^j reflects the oscillation clusters and runs about the diametrical axis passing through run $j - i/2$. The effect of the action δ^j is depicted graphically in Figure B.1.



(a) *Mapping of* δ^j *for even* i and $j = 1, \ldots, i/2$

(b) *Mapping of* δ^j *for even* i and $j = i/2 + 1, \ldots, i$

(c) *Mapping of* δ^j *for odd* i

FIGURE B.1: *The mappings of the action* δ^j *for (a)–(b) even* i *and (c) odd* i. *Each open square represents an oscillation cluster alternating between the substates* CDC *and* DDD, *while each solid circle represents a cooperation run of length at least 2.*

The group $\mathcal{V}$ is formed by the set $\{\iota, \rho^1, \rho^2, \ldots, \rho^{i-1}, \delta, \delta^2, \ldots, \delta^i\}$, and has order $2i$. According to the Cauchy-Frobenius Lemma,

$$\mathcal{U} = \sum_{i=1}^{\lfloor \frac{n}{5} \rfloor} \frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} |F_v|, \tag{B.2}$$

where F_v is the set of states that are fixed by action v.

The effect of the action ι is to map each element (oscillation cluster and cooperation run) to itself and therefore all possible states of the form (8.1) are fixed by this action. The $n - 5i$ cooperators can be distributed amongst i cooperation runs in

$$\binom{i + n - 5i - 1}{n - 5i} = \binom{n - 4i - 1}{i - 1}$$

ways. Having accounted for the distribution of cooperators, the remaining enumeration freedom lies in the selection of the phases of the oscillation clusters. Each oscillation cluster maps to itself and therefore there is complete freedom. Hence the total number of states fixed by the action ι is given by

$$|F_\iota| = 2^i \binom{n - 4i - 1}{i - 1}.$$

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The action ρ^j performs a modular shift of the elements by j positions to the right. Therefore the states in which the first j structures are free and subsequently repeated i/j times are fixed by the action ρ^j. It is important to note that in cases where i does not divide j, the action can only fix the states in which the first $\gcd(i, j)$ oscillation clusters and runs determine the rest. Furthermore, the number of cooperators to be distributed has to be sufficient to repeat the pattern decided upon $i/\gcd(i, j)$ times and this is possible when $i|n \gcd(i, j)$. Let $d = \gcd(i, j)$, then the number of states fixed by the action ρ^j is given by

$$|F_{\rho^j}| = 2^d \sum_{j\in\mathcal{T}} \binom{\frac{(n-5i)d}{i} + d - 1}{d-1},$$

where $\mathcal{T}$ is the set $\{x \in \mathbb{N} \mid i \text{ divides } n \gcd(i, x) \text{ and } x < i\}$.

The action δ^j performs a reflection about a diametrical axis which is handled slightly differently, depending on the parity of i. For even i and $j = 1, \ldots, i/2$, the axis in question passes through two oscillation clusters which may therefore be in either phase. The remaining oscillation clusters are mapped onto one another in pairs, resulting in $2^{\frac{i}{2}+2} = 2^{\frac{i+2}{2}}$ possible phase assignments of the oscillation clusters. The cooperation runs all map to one another in pairs, which means that assigning $\frac{n-5i}{2}$ cooperators among $i/2$ runs determines the distribution of all $n - 5i$ cooperators. Therefore the number of states fixed by all actions δ^j for even i and $j = 1, \ldots, i/2$ is given by

$$\sum_{j=1}^{\frac{i}{2}} |F_{\rho^j} = \sum_{j=1}^{\frac{i}{2}} 2^{\frac{i+2}{2}} \binom{\frac{i}{2} + \frac{n-5i}{2} - 1}{\frac{n-5i}{2}} = \frac{i}{2} 2^{\frac{i+2}{2}} \binom{\frac{n-4i}{2} - 1}{\frac{i}{2} - 1}. \tag{B.3}$$

For even i but $j = i/2 + 1, \ldots, i$, the axis of reflection runs through two cooperation runs which are therefore completely free, while the remaining $i - 2$ cooperation runs map onto one another in pairs. All of the i oscillation clusters map to one another in pairs and therefore there are $2^{\frac{i}{2}}$ phase assignments to the oscillation clusters. The distribution of p cooperators among $(i - 2)/2$ cooperation runs determines the distribution of $2p$ cooperators among $i - 2$ cooperation runs, and so the remaining $n - 5i - 2p$ cooperators may subsequently be distributed among the two completely free cooperation runs on the diametrical axis of reflection. The possible number of assignments of the $2p$ cooperators is given by

$$\binom{\frac{i-2}{2} + p - 1}{p}, \tag{B.4}$$

and subsequently the distribution of the remaining $n - 5i - 2p$ cooperators among the two free cooperation runs is given by

$$\binom{2 + n - 5i - 2p - 1}{n - 5i - 2p} = \binom{n - 5i - 2p + 1}{1} = (n - 5i - 2p + 1). \tag{B.5}$$

The value of p can range from 0 to $\lfloor \frac{n-5i}{2} \rfloor$ and therefore the number of states fixed by the action δ^j for even i and $j = i/2 + 1, \ldots, i$ is given by

$$\sum_{j=\frac{i}{2}+1}^{i} |F_{\delta^j}| = \frac{i}{2} 2^{\frac{i}{2}} \sum_{p=0}^{\lfloor \frac{n-5i}{2} \rfloor} \left[(n - 5i - 2p + 1) \binom{\frac{i-2}{2} + p - 1}{p} \right]. \tag{B.6}$$

Finally, for odd i, the action δ maps one oscillation cluster and one cooperation run to themselves while the remaining $i - 1$ oscillation clusters (and cooperation runs) map to one another in pairs.

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The freedom of phase in the oscillation clusters therefore accounts for an enumeration factor of $2^{\frac{i+1}{2}}$. Placing q cooperators among $(i-1)/2$ cooperation runs determines the distribution of $2q$ cooperators among $i-1$ cooperation runs. The remaining cooperators are subsequently distributed among the cooperation run lying on the axis of reflection. The number of states fixed by all of the δ^j actions for odd i is consequently given by

$$\sum_{j=1}^{i} |F_{\delta^j}| = \sum_{j=1}^{i} 2^{\frac{i+1}{2}} \sum_{q=0}^{\lfloor \frac{n-5i}{2} \rfloor} \binom{\frac{i-1}{2}+q-1}{q} = i \sum_{q=0}^{\lfloor \frac{n-5i}{2} \rfloor} 2^{\frac{i+1}{2}} \binom{\frac{i-1}{2}+q-1}{q}. \qquad \square$$